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AN L^1-SPACE FOR BOOLEAN ALGEBRAS AND SEMIREFLEXIVITY OF SPACES $L^\infty(X, \Sigma, \mu)$

BY

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ABSTRACT. In this paper we suggest a measure free construction of L^1-spaces using Boolean algebras and strict topologies and initiate a duality theory of (L^∞, L^1) like that of the duality of continuous functions and Baire measures, showing that the Boolean context yields a formal link between uniform tightness, uniform σ-additivity and uniform integrability.

1. **Introduction.** There has been a great deal of work in the past few years on (C_b, M) dual pairings of spaces C_b of bounded continuous functions and spaces M of bounded Baire, or regular Borel, or separable Baire or tight Borel measures defined on completely regular Hausdorff spaces; [3], [12], [26], [27] and [28] contain many references. The topologies of these dual pairings are far from normable, but quite workable versions of these have been found [26] as extensions of Buck's strict topology [2]. We wish to show that precisely the same kind of idea and method yields a workable dual topology for the pair (L^∞, L^1) and at the same time suggests an extended concept of L^1-spaces, "measure free" and directly dependent on the Boolean algebra of measurable sets. We will try to show that one can, using strict topologies, think of L^1 as the dual of L^∞ just as one thinks of M as the dual of C_b, and further, that one can in the same way think of the measures on any Boolean algebra $\mathcal{Q}$ as the strict dual of an appropriate real vector space $\mathbb{S}(\mathcal{Q})$ which carries the Boolean structure of $\mathcal{Q}$.

2. **Definitions and preliminaries.** A Boolean algebra $\mathcal{Q}$ is a commutative ring with identity e in which every element is multiplicatively idempotent, such as, for example, a ring of sets under intersection $\cap$ and symmetric difference $\triangle$. In $\mathcal{Q}$, it follows that $-a = a$ and that the definitions $a \wedge b = ab$ and $a \vee b = a + b + ab$ make $\mathcal{Q}$ into a distributive lattice. The definition $a \leqslant b$ iff $ab = a$ is a partial ordering of $\mathcal{Q}$ compatible with the lattice ordering and the definition of bounded subset, supremum and infimum are evident. $\mathcal{Q}$ is called complete if every subset of $\mathcal{Q}$ has a supremum in $\mathcal{Q}$ and σ-complete if

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every countable set has a supremum (for example, a σ-algebra of sets). We shall write $a_n \nearrow a$ ($a_n \searrow a$) in $\mathcal{A}$ to mean that $a_n \leqslant a_{n+1}$ ($a_n \geqslant a_{n+1}$) for each n, and a is the supremum (infimum) of the sequence $\{a_n\}$ in $\mathcal{A}$. More generally $\bigwedge_B b$ and $\bigvee_B b$ will denote the infimum and supremum respectively of a subset $B \subset \mathcal{A}$ when these exist. The elements a and b are disjoint in $\mathcal{A}$ iff $a \wedge b = 0$ and $\mathcal{A}$ is said to meet the countable chain condition (c.c.c.) if any pairwise disjoint subset of $\mathcal{A}$ is at most countable. We will need the theorem that if $\mathcal{A}$ is σ-complete and meets c.c.c., then $\mathcal{A}$ is complete, for with c.c.c. the supremum of $B \subset \mathcal{A}$ is always the supremum of a countable subset of B [16, p. 61]. Our basic references are Halmos [16] and Sikorski [29].

Our most important tool will be M. H. Stone's famous representation of $\mathcal{A}$ as the clopen (i.e., closed and open) subsets of a compact, Hausdorff topological space S. A ready definition of $S = S_{\mathcal{A}}$ is that it consist of all Boolean homomorphisms of $\mathcal{A}$ onto the simplest of Boolean algebras, $\{0, 1\}$ where $1 + 1 = 0$. As such S is a closed subset of the product $\mathbf{P}_{\mathcal{A}}\{0, 1\}$, hence compact. To represent $\mathcal{A}$ as an algebra of subsets of S, one corresponds a to the set $\eta(a) = \{h \in S; h(a) = 1\}$. This set is clearly open and closed in the relative product topology on S, hence the correspondence with clopen sets. But the important point is that by virtue of the compactness of S, every clopen set in S is some $\eta(a)$ [29, 8.2], and S is totally disconnected (distinct points are separated by disjoint clopen neighborhoods). It is a further result that the algebra $\mathcal{A}$ is complete iff S is extremally disconnected (the closure of every open set is open) [16, p. 92].

If $a_n \nearrow a$ in $\mathcal{A}$, then $\eta(a)$ is the closure in S of the union $\bigcup_{n=1}^{\infty} \eta(a_n)$ and it follows that $\eta(a) \backslash \bigcup_{n=1}^{\infty} \eta(a_n)$ is closed and nowhere dense in S. For example, if $\mathcal{A}$ is the algebra of finite/cofinite subsets of $\mathfrak{N} = \{1, 2, 3, \ldots\}$ and $a_n = \{1, 2, 3, \ldots, n\}$, consider the homomorphism h_0 such that h_0 is 0 on finite sets and 1 on cofinite sets. Then $S = \mathfrak{N} \cup \{h_0\}$ and S is the one point compactification of $\mathfrak{N}$, $a_n \nearrow a = \mathfrak{N}$, and $\{h_0\} = \eta(a) \backslash \bigcup_{n=1}^{\infty} \eta(a_n)$. Similar things can be said for increasing nets in the general $\mathcal{A}$ as well, and in general, $\eta(\bigvee_B b) = \overline{\bigcup_B \eta(b)}$ where the bar denotes closure.

In this paper we regard S as a transient tool in the study of $\mathcal{A}$ and its associated function, measure and functional spaces. We begin introducing these spaces with the real vector space $\mathbb{S}(\mathcal{A})$ of simple functions over $\mathcal{A}$. The collection of formal finite linear combinations of elements of $\mathcal{A}$ with real coefficients is a real-vector space which completely sheds the algebraic structure of $\mathcal{A}$. However each element $a \in \mathcal{A}$ corresponds to a continuous function $\chi(a) = \chi_{\eta(a)}$, the characteristic function of $\eta(a)$ in S. We denote by $\mathbb{S}(\mathcal{A})$ the set of all functions $\sum_{i=1}^{n} \alpha_i \chi(a_i)$, where α_i is real and $a_i \in \mathcal{A}$, and we will call the elements of $\mathbb{S}(\mathcal{A})$ simple functions over $\mathcal{A}$. $\mathbb{S}(\mathcal{A})$ is a real vector space under pointwise operations on these functions on S. We could of course dispense with S and remain purely algebraic, defining $\alpha a + \beta b = \alpha(a \backslash b)$

$+ \beta(b \backslash a) + (\alpha + \beta)ab$ where $a \backslash b = a - ab$ but much feeling would be lost. We denote by $\chi: \mathcal{A} \to \mathcal{S}(\mathcal{A})$ the crucial mapping $\chi(a) = \chi_{\eta(a)}$; note that $\chi(a)\chi(b) = \chi(ab)$ and $\chi(a + b) = \chi(a) + \chi(b)$ if $ab = 0$.

Now $\mathcal{S}(\mathcal{A}) \subset C(S)$, the continuous functions on S. In the uniform norm, $\|f\| = \sup\{|f(s)|: s \in S\}$, $C(S)$ is a Banach algebra, $\mathcal{S}(\mathcal{A})$ is a subalgebra which separates points of S and so its uniform closure is $C(S)$ by the Stone-Weierstrass theorem. For reasons to be yet given and consistent with [11] we denote the uniform closure of $\mathcal{S}(\mathcal{A})$ by $L^\infty(\mathcal{A}) = C(S)$. If $\{f_n\}$ is an increasing (decreasing) sequence of functions in $L^\infty(\mathcal{A})$ we write $f_n \nearrow$ ($f_n \searrow$) and we denote the lattice supremum (infimum) of $\{f_n\}$ in $L^\infty(\mathcal{A})$ by $\bigvee f_n$ ($\bigwedge f_n$), if it exists. By [11, 43D] $\mathcal{A}$ is σ-complete iff for every $f_n \nearrow$ in $L^\infty(\mathcal{A})$, $f = \bigvee f_n$ exists in $C(S)$; in general, $f(s) \geqslant \sup_n f_n(s)$. The example just above with $f_n = \chi(\{1, 2, \ldots, n\})$ offers a case of inequality (that $\mathcal{A}$ is not σ-complete is immaterial to this point). The corresponding result for complete algebras and increasing sets also holds but we make little direct use of this.

A finitely additive measure on $\mathcal{A}$ is a mapping $\mu: \mathcal{A} \to R$ (the reals) such that $\mu(a + b) = \mu(a) + \mu(b)$ if $ab = 0$. A positive measure on $\mathcal{A}$ is a nonnegative μ such that $a_n \nearrow a$ in $\mathcal{A}$ implies $\mu(a_n) \to \mu(a)$ (μ is countably additive). A measure on $\mathcal{A}$ is the difference of two positive measures on $\mathcal{A}$. Since $\mu(e) < \infty$ for μ positive, measures are always finite valued. The following is well known.

PROPOSITION 2.1. *There is a* 1-1 *correspondence between bounded linear functionals* $\hat{\mu}$ *on* $L^\infty(\mathcal{A})$ *and finitely additive measures* μ *on* $\mathcal{A}$ *such that* $\sup\{|\mu(a)|: a \in \mathcal{A}\} < \infty$, *given by* $\mu = \hat{\mu} \circ \chi$.

For if $ab = 0$, then $\chi(a + b) = \chi(a) + \chi(b)$, and the formula $\mu = \hat{\mu} \circ \chi$, then defines a finitely additive measure on $\mathcal{A}$ such that $\sup\{|\mu(a)|: a \in \mathcal{A}\} \leqslant \|\hat{\mu}\| = \sup\{|\hat{\mu}(f)|: \|f\| \leqslant 1\} < \infty$. Conversely, if μ is given then $\hat{\mu}$ defined on $\mathcal{S}(\mathcal{A})$ by $\hat{\mu}(\sum_{i=1}^n \alpha_i \chi(a_i)) = \sum_{i=1}^n \alpha_i \mu(a_i)$ is well defined and uniformly bounded for $\|f\| \leqslant 1$ by $\sup\{|\mu(a)|: a \in \mathcal{A}\}$. Because $\mathcal{S}(\mathcal{A})$ is $\|\ \|$-dense in $L^\infty(\mathcal{A})$, $\hat{\mu}$ has a unique bounded extension to a linear functional on $L^\infty(\mathcal{A})$.

Note that for the example of $\mathcal{A} = \mathfrak{N}$ above and the functional $\hat{\mu}(f) = f(h_0)$, the representing measure is not countably additive on $\mathcal{A}$. Like the uniform norm dual of spaces $C_b(X)$, X not compact, or infinity norm dual of spaces $L^\infty(X, \Sigma, \mu)$, the norm dual of $L^\infty(\mathcal{A})$ is not measure theoretically satisfying.

This measure theoretic defect in the dual representation (Riesz representation theorem) of linear functionals on $C_b(X)$ was first remedied in the locally compact case by R. C. Buck [2] who defined the strict topology β on $C_b(X)$ as that locally convex topology on $C_b(X)$ defined by the seminorms $\|f\|_\xi = \sup\{|f(x)\xi(x)|: x \in X\}$ where $f \in C_b(X)$ and $\xi \in C_b(X)$ vanishes at infinity. The strict dual of $C_b(X)$ is the space $M_t(X)$ of bounded Borel measures μ on X which are compact regular and in particular tight: $|\mu|(X)$

$= \sup\{|\mu|(K)\colon K$ compact in $X\}$, where $|\mu|$ is the total variation of μ. This setting was abstracted to that of Banach B-modules X by D. C. Taylor and the author [27] and subsequently used in the study of C^*-algebras.

For completely regular X, this writer [26] extended Buck's ideas to obtain a duality theory for the pairing (C_b, M) where M represents any one of the classes M_t, M_τ or M_σ of tight, τ-additive or σ-additive functionals of Varadarajan [30]. The method is tied to the definition of a strict topology β on $C_b(X)$ as the inductive limit of strict topologies β_Q defined by seminorms $\|f\|_\xi$, where now ξ is defined on the Stone-Čech compactification βX of X and vanishing on the compact set $Q \subset \beta X \backslash X$, the inductive limit being taken over all such Q. The idea unifying (L^∞, L^1)-duality and (C_b, M)-duality which we suggest is to replace βX by a Stone space S and the sets Q by appropriate subsets of S. That itself is not new for the ordinary L^∞ spaces of measure theory, for in [17] Henry and Taylor used the ideas in [26] and algebraic ideals rather than sets, to define a strict topology $\bar{\beta}$ for W^*-algebras and obtain the predual as the $\bar{\beta}$-dual, and of course ordinary L^∞ is a W^*-algebra with predual L^1. What we offer herein is a β-theory that is immediately and simply tied to the defining notion of measure theory, countable additivity, and which, as it develops, suggests an extended idea of L^1-spaces. At the same time, our β-theory simultaneously offers a dual treatment of bounded Borel functions and bounded Borel measures whose basic theorems are suggested by (L^∞, L^1).

Returning to the issue of countably additive measures on $\mathcal{Q}$ we want to include a known preliminary result [11] which simplifies some technical details in the sequel. If $\hat{\mu}$ is a bounded linear functional on $L^\infty(\mathcal{Q})$ we can write $\hat{\mu} = \hat{\mu}^+ - \hat{\mu}^-$ where for $f \geqslant 0$

$$\hat{\mu}^+(f) = \sup\{\hat{\mu}(g)\colon 0 \leqslant g \leqslant f\}$$

and $\hat{\mu}^- = \hat{\mu}^+ - \hat{\mu}$, and $\hat{\mu}^+$ are bounded functionals on $L^\infty(\mathcal{Q})$.

Proposition 2.2. *If $\hat{\mu}(\chi(a_n)) \to 0$ for $a_n \searrow 0$ in $\mathcal{Q}$, then $\hat{\mu}^+ \circ \chi$ is a positive measure on $\mathcal{Q}$.*

Proof. Let $\mu = \hat{\mu} \circ \chi$ and $\mu^+ = \hat{\mu}^+ \circ \chi$. We claim that $\mu^+(a) = \sup\{\mu(b)\colon b \leqslant a\}$ for each $a \in \mathcal{Q}$. Clearly $\mu^+(a) \geqslant \sup\{\hat{\mu}(\chi(b))\colon b \leqslant a\} = \sup\{\mu(b)\colon b \leqslant a\}$, and $\mu^+(a) = \sup\{\hat{\mu}(f)\colon 0 \leqslant f \leqslant \chi(a)\}$. If $0 \leqslant f \leqslant \chi(a)$, we can choose a sequence of simple functions $f_n \in \mathbb{S}(\mathcal{Q})$ such that

$$\|f_n - f\| \geqslant \|(f_n - f)\chi(a)\| \to 0.$$

From this we can suppose $f_n = \sum_{i=1}^{k_n} \alpha_i^n \chi(b_i^n)$ with $0 \leqslant \alpha_i^n \leqslant 1$ and $b_i^n \leqslant a$. Letting $b_n = \max\{b_i^n\colon \mu(b_i^n) \geqslant 0\}$ we obtain $\hat{\mu}(f) \leqslant \lim_n \mu(b_n)$ because $\hat{\mu}$ is bounded. Hence $\mu^+(a) = \sup\{\mu(b)\colon b \leqslant a\}$.

Now we have to show that $\mu^+(a_n) \to 0$ if $a_n \searrow 0$ in $\mathcal{Q}$. If $\mu^+(a_n) \geqslant \varepsilon$ for all

n, we can find a $b \in \mathcal{A}$, $b \leqslant a_1$, such that $\mu(b) > \varepsilon/2$. Since $ba_n \searrow 0$ there is an $n_1 > 1$ such that $|\mu(ba_{n_1})| < \mu(b) - \varepsilon/2$. Then

$$\mu^+(a_1) - \mu^+(a_{n_1}) = \mu^+(a_1 - a_1 a_{n_1}) \geqslant \mu(b - ba_{n_1}) > \varepsilon/2.$$

Inductively defining $\{a_{n_i}\}$ in this fashion we contradict the fact that $\sum_{n=1}^{\infty} (\mu^+(a_n) - \mu^+(a_{n+1})) < \infty$. Hence $\mu^+ = \hat{\mu}^+ \circ \chi$ is a measure on $\mathcal{A}$.

We will in the sequel refer to the idea of a Boolean measure algebra. The definition is in §8, so we will not repeat it, but the model is the complete Boolean algebra $\mathcal{A}$ of Lebesgue measurable sets in [0, 1] mod sets of measure zero with the measure induced on $\mathcal{A}$ (as in 4.7) by Lebesgue measure. The duality theory for these has especially nice properties (§8) and our earlier sections are pointed towards exposing the degree to which any Boolean algebra acts like a Boolean measure algebra in the sense of its natural classes of (simple) functions and measures.

Our terminology for locally convex topological vector space theory is drawn from [22]; in particular if E and E' are in duality we denote the action of elements on each other by $\langle x, x' \rangle$, $x \in E$, $x' \in E'$ and we denote the weak topology of E' on E by $\sigma(E, E')$ and the weak* topology of E' on E by $\sigma(E', E)$. For ordered vector space theory we use [20] and sometimes [23]. If T is a locally convex topology on E, $W \in T$ means that W is a T neighborhood of zero.

In closing these preliminaries I would like to take special note of other work, which though seldom directly referenced, has been indispensable to the formation of these ideas we present. The first was due to S. Scheinberg [24] who retopologized the real line in a completely regular fashion so as to obtain the Lebesgue sets as the Borels for this new topology and the L^∞-functions as bounded and continuous. The utter failure of the strict topologies of [26] to say anything about L^1 in this setting suggested that something was being missed. Subsequently J. Dazord and M. Jourlin [6] obtained L^1 as an L^∞-dual for a tight Borel measure by a localization topology similar to β_0 [26] with convergence in measure replacing the compact-open topology. Henry and Taylor [17] improved and extended this to W^*-algebras with L^1 already at hand. The key idea of §3 defining a strict topology for $L^\infty(\mathcal{A})$ was drawn from a preprint by Graves and Molnar [15] along with their "gliding hump" argument which underlies the important result 4.9 herein. Once things began to come together, the work of D. H. Fremlin [11] provided an inestimable impetus to develop the initial ideas much further. Finally I would like to thank my good friend Bob Wheeler (of [28]) for his generous and steady encouragement in the initiation and completion of this work.

A word on methodology. We want to study measurable functions as continuous functions and use as much topology as needed. In the same vein we make free use of the Riesz representation theorem for $C(S)$ if its use simplifies arguments about the algebra $\mathcal{Q}$. A dual treatment of (L^∞, L^1) rather than measure theory *per se* is the goal, though many arguments are clearly reminiscent of standard measure theoretic arguments. We wish to show that we have a tenable extension and a viable format for studying L^1- and L^∞-spaces without proving too many generalizations of known results about ordinary L^1-spaces.

3. **A strict topology for spaces** L^∞. There is one setting where L^∞ is immediately a space $C(S)$, and it is the only setting in which a strict topology treatment of (l^∞, l^1) has been done. If D is a set with discrete topology, then D is locally compact, $l^\infty(D) = C_b(D)$ and from Buck's original work [2], the strict topology β on $l^\infty(D)$ yields $M_t(D) = l^1(D)$ as dual to L^∞. A detailed study of l^∞_β can be found in [4]. In particular, $\sigma(l^1, l^\infty)$-countably compact sets are β-equicontinuous and Conway's [5] criteria for β-equicontinuity (uniform tightness) is the Dunford-Pettis criteria for weakly compact subsets of l^1 (whose norm dual is l^∞). In this section we define β for any L^∞-space and in later sections go on to show that this important result holds in this generality.

Let $\{a_k\}$ be an increasing sequence in $\mathcal{Q}$ with supremum $a \in \mathcal{Q}$. Let $Q = \eta(a)\backslash\bigcup_{k=1}^\infty \eta(a_k)$ in S; Q is compact and nowhere dense. Let $C_Q = C_0(S\backslash Q) = \{\xi \in C(S): \xi \equiv 0 \text{ on } Q\}$. Then C_Q is a Banach subalgebra of $C(S)$ with countable approximate identity $\xi_k = \chi(a_k) + \chi(e - a) = \chi_{\eta(a_k)} + \chi_{S\backslash\eta(a)}$, and $C(S)$ is a C_Q module. Using [27], we define the strict topology β_Q on $L^\infty(\mathcal{Q}) = C(S)$ by the seminorms

$$\|f\|_\xi = \sup\{|f(x)\xi(x)|: x \in S\},$$

one for each $\xi \in C_Q$. It suffices to use $\xi \geqslant 0$. Now, following [26], we define the strict topology β on $L^\infty(\mathcal{Q})$ as the inductive limit of the topologies β_Q taken over all such Q. An absolutely convex absorbent set W is a β neighborhood of 0 iff W is a β_Q neighborhood for each such Q, we denote this by $W \in \beta$. A neighborhood base for β at 0 can be made to consist of convex hulls of sets of the form $\bigcup V_Q$ where $V_Q \in \beta_Q$ and the union ranges of all such Q.

In relation to strict topologies on spaces $C_b(X)$ for completely regular X it will become clear in the sequel that β herein is the analogue of β_1 in [26], but we will omit the subscript. We want to also define the topology $\bar{\beta}$ on $L^\infty(\mathcal{Q})$ as the inductive limit of the topologies β_Q where this time Q ranges over all sets $\eta(a)\backslash\bigcup_\alpha \eta(a_\alpha)$ where $a_\alpha \nearrow a$ in $\mathcal{Q}$. The topology $\bar{\beta}$ is the analogue of Henry and Taylor's $\bar{\beta}$ for W^*-algebras [17] and of β in [26]. For complete Boolean algebras $\bar{\beta}$ has advantages which we make use of in §6. We will concentrate herein on β for a number of reasons: (1) The ready connection with countable additivity. (2) Subsequent results. (3) Application to σ-algebras of sets and bounded measurable functions where the appropriate $\mathcal{Q}$ is not in general complete. At the same time a number of our results hold simultaneously for β and $\overline{\beta}$ and we will so note these. Finally, for $X = \mathfrak{N}$ and $\mathcal{Q} = 2^{\mathfrak{N}}$ it is easy to see that $S_{\mathcal{Q}} = \beta\mathfrak{N}$ and that $\beta = \beta_1 = \overline{\beta}$ is Buck's β on $C(S) = l^\infty = L^\infty(\mathcal{Q})$.

We complete this section with another example and then list three initial results basic to the remainder of this paper.

Firstly, and unlike the usual strict topologies [26, 2.1], β is neither Hausdorff nor stronger than pointwise convergence on S, in general. For if $\mathcal{Q}$ is the Boolean algebra of finite/cofinite subsets of $\mathfrak{N}$ of §2 then $\chi(a_n) \to \chi(a)$ in the topology β (we will write $\chi(a_n) \xrightarrow{\beta} \chi(a)$), but $\chi(a_n)(h_0) \nrightarrow \chi(a)(h_0)$ where $a_n = \{1, 2, \ldots, n\}$. Secondly, we relate β to the usual supremum norm topology $\| \ \|$ on $C(S)$.

Proposition 3.1. (a) $\beta \leqslant \| \ \|$ *on* $L^\infty(\mathcal{Q})$.

(b) $\beta = \| \ \|$ *iff every increasing* $\{a_n\} \subset \mathcal{Q}$ *with supremum in* $\mathcal{Q}$ *is finite.*

Proof. (a) For, the empty set is a set Q defining β.

(b) If $\beta = \| \ \|$, then $B_1 = \{f \colon \|f\| \leqslant 1\} \in \beta_Q$ for any Q. Let $Q = \eta(a)\backslash\bigcup_k \eta(a_k)$. Then there is a $\xi \in C_0(Q)$ such that $\|f\|_\xi \leqslant 1$ implies $f \in B_1$. Hence $\min(1/\xi, n) \in B_1$ for any n, where $1/\xi \in B_1$ since B_1 is pointwise closed. This means that $\xi \geqslant \varepsilon > 0$ on all of $S\backslash Q$ making $S\backslash Q$ compact and hence covered by $S\backslash\eta(a)$ and $\eta(a_n)$ for some n. But $k \geqslant n$ puts $\eta(a_k) \subset \eta(a_n)$ which makes $a_k \leqslant a_n \leqslant a_k$ and hence $a_k = a_n$.

Conversely, if only finite increasing sequences admit a supremum, this supremum is the maximum element, and every set Q is empty, whence $\beta = \| \ \|$.

The three theorems which follow are stated more generally and underlie the most pleasant and useful properties of strict topologies defined through an inductive limit, and without which these topologies would be difficult to deal with.

Let S be any compact Hausdorff space and let $\mathfrak{Q}$ be a fixed collection of closed subsets of S such that $S\backslash Q$ is dense in S for each $Q \in \mathfrak{Q}$. Let the β_Q topology be defined on $C(S)$ by seminorms $\|f\|_\zeta$ where $\zeta \in C_0(S\backslash Q)$ just as above; each β_Q is Hausdorff. Let β be the inductive limit over all such Q. The first result was proven initially by Dorroh [8], see also [25]. Let $B_r = \{f \in C(S) \colon \|f\| \leqslant r\}$.

Theorem 3.2. *β is the finest locally convex topology on $C(S)$ agreeing with itself on each set B_r for all $r > 0$. Consequently, the β-continuity of linear maps is determined solely on the sets B_r.*

The proof is exactly as in [26, 3.1].

A bounded linear functional ϕ on the Banach space $C(S)$ can be written $\phi = \phi^+ - \phi^-$ where $\phi^\pm(f) \geqslant 0$ for all $f \geqslant 0$ and $\phi^\pm$ is bounded. Let $|\phi| = \phi^+ + \phi^-$. Exactly as in [26, 4.2] we have

THEOREM 3.3. *The β-continuity of ϕ is equivalent to the β-continuity of $|\phi|$ or of ϕ^+ and ϕ^-.*

Asymptotic results in the theory of probability obtained through the study of weak (or weak*) convergence often involve an interchange of limits of function f and measure P in a dual bilinear form $\langle f, P\rangle = \int_R f dPX^{-1} = \int_\Omega f \circ X\, dP$ [1]. The joint continuity of $(f, P) \to \langle f, P\rangle$ becomes more likely if the domains are compact in some sense. The usual compactness criteria is equicontinuity of the domain H of P relative to the topology defining convergence for the domain of $f \circ X$. If H consists of tight Borel measures this equicontinuity is uniform tightness: There is a compact K such that $P(\Omega\backslash K) < \varepsilon$ uniformly over $P \in H$. But this is the Conway-LeCam criteria for β-equicontinuity [5]. On the other hand, if each PX^{-1}, $P \in H$, has a density in some space $L^1(m)$, this equicontinuity becomes uniform integrability (i.e., the conditions of the Dunford-Pettis theorem for weak compactness in $L^1(m)$) and uniform integrability and uniform tightness have been informally considered as the same. We will show that they are formally the same and both are β-equicontinuity, the former for our space $L^1(\mathcal{Q})$ (to be defined), the latter for the pairing (C_b, M_t) having already been obtained [12], [26]. This is the significance of the next result which ultimately rests on a characterization of β-equicontinuity in terms of approximate identities [27] for spaces $C_0(X)$, or in our case $C_0(S\backslash Q)$. Loosely, the result means that strict topologies ignore behavior at infinity while at the same time not ignoring behavior near infinity, infinity being played by each set Q.

THEOREM 3.4. *A set H of β-continuous linear functionals on $C(S)$ is β-equicontinuous iff*

(a) *H is bounded; i.e.* $\sup_{\phi\in H}|\phi|(1) < \infty$ *and*

(b) *H is uniformly tight for each $Q \in \mathfrak{Q}$ at a time; i.e. given $Q \in \mathfrak{Q}$ and $\varepsilon > 0$ and $r > 0$ there is a compact subset $K \subset S\backslash Q$ such that $\|f\| \leqslant r$ and* $\sup\{|f(x)|: x \in K\} \leqslant 1$ *implies $|\phi(f)| < \varepsilon$ for all $\phi \in H$.*

If we represent each $\phi \in H$ by $\phi(f) = \int_S f d\mu^*$, μ^* a regular Borel measure on S, condition (b) says that $|\mu^*|((S\backslash Q)\backslash K) < \varepsilon$ for all $\phi \in H$; this condition is usually referred to as "uniform tightness".

For a proof of 3.4 note that, equicontinuity for an inductive limit β is equicontinuity for each Q at a time [22] and (a) and (b) characterize β_Q-equicontinuity ([5], [27], or [26]).

4. The space $L^1(\mathcal{Q})$ and the vector measure χ. Although the analogy is not entirely correct, we define $L^1(\mathcal{Q})$ as the β-dual of $L^\infty(\mathcal{Q})$. In this section we begin to justify the name, and in later sections see that some of the more useful applications of this measure free construction of an L^1-space are true analogues of the ordinary L^1-spaces of measure theory.

The basis of this paper is the simple result, 4.1, below. It, along with 4.3, are the reasons for our definition of β. We have already noted that χ: $\mathcal{C} \to L^\infty(\mathcal{C})$ is a finitely-additive vector valued function on $\mathcal{C}$. If $a_n \nearrow a$ in $\mathcal{C}$, $\{\chi(a_n)\}$ utterly fails to converge in the norm topology to $\chi(a)$. This is akin to the failure of $L^1(X, \Sigma, \mu)$ to result in the infinity norm dual of $L^\infty(X, \Sigma, \mu)$.

THEOREM 4.1. *χ: $\mathcal{C} \to (L^\infty(\mathcal{C}), \beta)$ is σ-additive. That is, if $a_n \nearrow a$ in $\mathcal{C}$, then* $\chi(a_n) \to^\beta \chi(a)$.

PROOF. Suppose $a_n \nearrow a$ in $\mathcal{C}$. Let $W \in \beta$. Then for $Q = \eta(a)\backslash\bigcup_{k=1}^\infty \eta(a_k)$, $W \in \beta_Q$ and hence there is a $\xi \in C_Q$ such that $\|f\xi\| \leqslant \varepsilon$ puts $f \in W$.

The set $K = \{x \in S\colon \xi(x) \geqslant \varepsilon/2\}$ is compact and covered by $S\backslash\eta(a)$ and $\{\eta(a_k)\colon k = 1, 2, \ldots\}$. Hence there is an N such that $K \subset S\backslash\eta(a) \cup \eta(a_N)$. If $n \geqslant N$, then $\chi(a) - \chi(a_n) \neq 0$ only on the set $\eta(a)\backslash\eta(a_n) \subset S\backslash K$, where $\xi < \varepsilon/2$. Hence $\|(\chi(a) - \chi(a_n))\xi\| \leqslant \varepsilon$, whence $\chi(a) - \chi(a_n) \in W$ for $n \geqslant N$.

COROLLARY 4.2. *If $\{b_n\}$ is a pairwise disjoint sequence in $\mathcal{C}$, then*

$$\sum_{n=1}^\infty \chi(b_n) \overset{\beta}{=} \chi\Big(\bigvee_n b_n\Big).$$

Measures on Boolean algebras have long been of interest. We let $M(\mathcal{C})$ denote the real vector space of measures on $\mathcal{C}$ (as defined in §2). There is something in a name and as this work developed it became evident that the β-dual of $L^\infty(\mathcal{C})$ has two personalities. When $\mathcal{C}$ is complete we will see that the β-dual $L^1(\mathcal{C})$ does behave like an ordinary L^1-space, its norm dual being $L^\infty(\mathcal{C})$ in most cases. When $\mathcal{C}$ is, say, only σ-complete, then $L^1(\mathcal{C})$ behaves more like the Banach space of bounded measures on a σ-algebra, more like what one would call $M(\mathcal{C})$, and its norm dual in these cases is not $L^\infty(\mathcal{C})$. Wishing to use a single notation we choose L^1. We also note that D. H. Fremlin [11] has suggested several abstractions of L^1-spaces using Riesz space theory, the most notable being $L^\#(\mathcal{C})$ [11, 44A], in general a proper subset of our $L^1(\mathcal{C})$. Our second fundamental result characterizes β-continuous functionals as measures on $\mathcal{C}$, that is, members of $M(\mathcal{C})$.

THEOREM 4.3. *The formula $\mu = \hat{\mu} \circ \chi$ defines a one-to-one, onto correspondence between the positive measures on $\mathcal{C}$ and the positive linear functionals $\hat{\mu} \in L^1(\mathcal{C})$.*

PROOF. By 4.1, $\mu(a) = (\hat{\mu} \circ \chi)(a)$ defines a positive measure on $\mathcal{C}$ if $\hat{\mu} \in L^1(\mathcal{C})$ and $\hat{\mu}$ is positive.

Conversely, given a positive measure on $\mathcal{C}$ consider the bounded linear functional $\hat{\mu}$ on $L^\infty(\mathcal{C})$ defined in 2.1 by the measure μ according to $\mu = \hat{\mu} \circ \chi$. We will show that $\hat{\mu}$ is β-continuous.

Let $W = \hat{\mu}^{-1}(-3\varepsilon, 3\varepsilon)$. We claim $W \in \beta$. Given $Q = \eta(a)\backslash\bigcup_{k=1}^\infty \eta(a_k)$ it suffices to find for each $r \geqslant 1$ a $\xi \in C_Q$ such that $\|f\|_\xi \leqslant 1$ and $f \in B_r$ puts $f \in W$.

Since μ is a measure we can find an n_0 such that $\mu(a) - \mu(a_n) < \varepsilon/r$ for $n \geqslant n_0$. Let $\xi = \chi(e - a) + \chi(a_n)$ and suppose $\|f\|_\xi \leqslant \varepsilon/\mu(e)$ where $f \in B_r \cap S(\mathcal{Q})$, $f = \sum_{i=1}^p \alpha_i \chi(b_i)$ with $b_i b_j = 0$ for $i \neq j$. Then,

$$|\hat{\mu}(f)| = \left|\sum_{i=1}^p \alpha_i \mu(b_i)\right|$$
$$\leqslant \sum_{i=1}^p |\alpha_i| \mu(b_i a_n) + \sum_{i=1}^p |\alpha_i| \mu(b_i(e + a)) + \sum_{i=1}^p |\alpha_i| \mu(a + a_n)$$

recalling that $e = a_n + (e + a) + (a + a_n)$ and $a + b = a - b$ in $\mathcal{Q}$.

Now

$$\sum_{i=1}^p |\alpha_i| \mu(b_i(a + a_n)) \leqslant \sum_{i=1}^p r\mu(b_i(a + a_n))$$
$$\leqslant r\mu\left(\sum_{i=1}^p b_i(a + a_n)\right) \leqslant r\mu(a + a_n) \leqslant r(\varepsilon/r) = \varepsilon.$$

Also, if $\mu(b_i a_n) \neq 0$, then there is an $x \in \eta(b_{i_p}) \cap \eta(a_n)$, whence

$$|\alpha_i| = |\alpha_i \xi(x)| \leqslant \varepsilon/r$$

and

$$\sum_{i=1}^P |\alpha_i| \mu(b_i a_n) \leqslant \frac{\varepsilon}{r} \sum_{i=1}^P \mu(b_i a_n) \leqslant \frac{\varepsilon}{r} \mu(a_n) \leqslant \varepsilon.$$

Similarly, $\sum_{i=1}^p |\alpha_i| \mu(e + a) \leqslant \varepsilon$. Hence $|\hat{\mu}(f)| \leqslant 3\varepsilon$ on $B_r \cap \mathcal{S}(\mathcal{Q})$. But B_r is the norm closure of $B_r \cap \mathcal{S}(\mathcal{Q})$ whence $|\hat{\mu}(f)| \leqslant 3\varepsilon$ on B_r, so that $W \in \beta_Q$. Observing that $\mu(a) = (\hat{\mu} \circ \chi)(a)$, we have a proof.

Corollary 4.4. *For a bounded linear functional $\hat{\mu}$ on $L^\infty(\mathcal{Q})$ these are equivalent*:

(a) *$\hat{\mu}$ is β-continuous.*

(b) *$|\hat{\mu}| \circ \chi$ is a measure on $\mathcal{Q}$.*

(c) *If $a_n \searrow 0$ in $\mathcal{Q}$, then $\hat{\mu}(\chi(a_n)) \to 0$.*

Proof. Clearly (a) $\to$ (c). Using 2.2, (c) implies that $\mu^\pm \circ \chi$ are positive measures on $\mathcal{Q}$ and hence (c) $\to$ (b). Now using 4.3, (b) implies $|\hat{\mu}|$ is β-continuous and (a) follows from 3.3.

Since $\mathcal{S}(\mathcal{Q})$ is β-dense in $L^\infty(a)$ by 3.1(a), the uniqueness of the relation $\mu = \hat{\mu} \circ \chi$ implies

Corollary 4.5. *The β-dual of $\mathcal{S}(\mathcal{Q})$ is $M(\mathcal{Q})$.*

We will retain throughout the notations μ and $\hat{\mu}$, $\mu = \hat{\mu} \circ \chi$ for corresponding functionals on $L^\infty(\mathcal{Q})$ and measures on $\mathcal{Q}$. An interesting application of 4.3 yields another non-Hausdorff β and an interesting $L^1(\mathcal{Q})$. Let $\mathcal{Q}$ be the algebra of Borel sets of [0, 1] mod the ideal of meager Borel sets. It is an old result [16, p. 68] that there are no nonzero measures on $\mathcal{Q}$, hence $L^1(\mathcal{Q}) = \{0\}$ even though $L^\infty(\mathcal{Q})$ is large.

We turn now to a characterization of functionals $\hat{\mu}$ in terms of the Stone space S, analogous to the description of strictly continuous functionals on $C_b(X)$-spaces [26, Theorems 1.1 and 4.4]. By 3.1(a), a β-continuous functional $\hat{\mu}$ is $\| \ \|$-bounded on $L^\infty(\mathcal{Q}) = C(S)$; hence, there is a unique regular Borel measure μ^* on S such that $\hat{\mu}(f) = \int_S f\,d\mu^*$ for all $f \in L^\infty$.

THEOREM 4.6. *Let μ^* be a nonnegative Borel measure on S. The formula $\hat{\mu}(f) = \int_S f\,d\mu^*$ defines a β-continuous functional on $L^\infty(\mathcal{Q})$ iff $\mu^*(S) < \infty$ and μ^* vanishes on all sets $Q = \eta(a)\backslash\bigcup_{n=1}^\infty \eta(a_n)$.*

PROOF. By 4.1, if $Q = \eta(a)\backslash\bigcup_{n=1}^\infty \eta(a_n)$, then $\hat{\mu}(\chi(a_n)) \to_n \hat{\mu}(\chi(a))$ if $\hat{\mu}$ is β-continuous. But this means $\mu^*(\eta(a_n)) \to \mu^*(\eta(a))$, whence $\mu^*(Q) = 0$. Conversely, if $\mu^*(Q) = 0$, then $\mu^*(\eta(a)) = \lim_n \mu^*(\eta(a_n))$ which means that $\hat{\mu}(\chi(a_n)) \to \hat{\mu}(\chi(a))$, making $\hat{\mu}$ β_Q-continuous.

One reason for a role in measure theory for Boolean algebras are longstanding results identifying bounded measurable functions with continuous functions on the Stone space, and equivalence classes of measurable sets with elements of the appropriate algebra [29]. I know of no results which simply and directly carry this transformation through to a construction of ordinary L^1-spaces. We will see how this can be done and see where the *appelage* $L^1(\mathcal{Q})$ originates.

Let $(X, \Sigma, \bar{\mu})$ be a σ-finite measure space and let $\mathcal{Q}$ be Σ mod the $\cap$, $\triangle$ ideal $\bar{\mu}^{-1}(0)$. Let [ ] denote equivalence classes in $\mathcal{Q}$ or in $L^\infty(X, \Sigma, \bar{\mu})$ alternatively. Now define θ: $L^\infty(X, \Sigma, \bar{\mu}) \to L^\infty(\mathcal{Q})$ as follows: Let $\theta[\chi_E] = \chi([E])$ where $E \in \Sigma$, extend θ linearly to equivalence classes of simple functions in $L^\infty(X, \Sigma, \bar{\mu})$ and note that θ is an $\| \ \|_\infty - \| \ \|$ isometry and so extends uniquely to an isometry from $L^\infty(\bar{\mu}) = L^\infty(X, \Sigma, \bar{\mu})$ onto $L^\infty(\mathcal{Q})$. Thus we can identify $L^\infty(\bar{\mu})$ with $L^\infty(\mathcal{Q}) = C(S)$. In fact, there is a formula giving θ directly [29, p. 155]

$$\theta[f](h) = \sup\{r: h([f \geqslant r]) = 1\}$$

where $h \in S$ is a homomorphism of $\mathcal{Q}$.

THEOREM 4.7. *The adjoint mapping θ' maps $L^1(\mathcal{Q})$* 1-1 *and onto $L^1(X, \Sigma, \bar{\mu})$ and is a isometry under the norm $\|\hat{\mu}\|_1 = \sup\{|\hat{\mu}(f)|: \|f\| \leqslant 1, f \in L^\infty(\mathcal{Q})\}$ for $\hat{\mu} \in L^1(\mathcal{Q})$.*

PROOF. The proof is quite apparent with the proper identifications. First we consider the elements of $L^1(X,\Sigma,\bar{\mu}) = L^1(\bar{\mu})$ as measures on Σ absolutely continuous with respect to $\bar{\mu}$. Then we identity $\bar{\nu}$ with the functional $\bar{\nu}[f] = \int_X f d\bar{\nu}$ for $[f] \in L^\infty(\bar{\mu})$. Hence we have $L^1(\bar{\mu}) \subset (L^\infty(\bar{\mu}), \| \|_\infty)'$ and this is an isometric imbedding as well.

By 3.1(a), $L^1(\mathcal{Q}) \subset (L^\infty(\mathcal{Q}), \| \|)'$ and because θ is an $\| \|_\infty - \| \|$ isometry, θ' is an isometric imbedding of $L^1(\mathcal{Q})$ in $(L^\infty(\bar{\mu}), \| \|_\infty)'$ under the norm, $\| \|_1$, defined above. Observe also that the functional $\bar{\nu}$ defined above is completely determined by its values at points $[\chi_E] \in L^\infty(\bar{\mu})$.

To complete the argument, first suppose that $\bar{\nu} \in L^1(\bar{\mu})$. By absolute continuity, $\nu[E] = \bar{\nu}(E)$ is a well-defined measure on $\mathcal{Q}$. By 4.3, $\nu = \hat{\nu} \circ \chi$ for some $\hat{\nu} \in L^1(\mathcal{Q})$. But we then have

$$\bar{\nu}[\chi_E] = \nu(E) = (\hat{\nu} \circ \chi)[E] = \hat{\nu}(\chi[E]) = \hat{\nu}(\theta[\chi_E]) = \theta'\hat{\nu}[\chi_E].$$

Hence $L^1(\bar{\mu}) \subset \theta L^1(\mathcal{Q})$. Conversely, given $\hat{\nu} \in L^1(\mathcal{Q})$, the formula $\bar{\nu}(E) = (\hat{\nu} \circ \chi)(E)$ defines a measure on Σ absolutely continuous with respect to $\bar{\mu}$, completing the proof.

Thus we have in this sense a measure-free construction of $L^1(\bar{\mu})$, namely as $L^1(\mathcal{Q})$, directly dependent only on Σ and the ideal $\bar{\mu}^{-1}(0)$. With the next theorem we begin the direct construction of $L^1(\bar{\mu})$ as the dual of $L^\infty(\bar{\mu})$ under what will, in §8, prove to be a familiar topology, definable without passage through S, $L^\infty(\mathcal{Q})$ and $L^1(\mathcal{Q})$.

We define the β_∞ topology on $L^\infty(\bar{\mu})$ as the weakest topology making θ continuous. Because θ is 1-1 and onto, θ is a β_∞-β topological isomorphism and,

THEOREM 4.8. *$L^1(\bar{\mu})$ is the β_∞-dual of $L^\infty(\bar{\mu})$. Moreover, β_∞ is the finest locally convex topology on $L^\infty(\bar{\mu})$ with dual $L^1(\bar{\mu})$. Indeed, any $\sigma(L^1(\bar{\mu}), L^\infty(\bar{\mu}))$-countably compact set in $L^1(\bar{\mu})$ is β_∞-equicontinuous.*

The first statement is apparent from 4.7. The latter two statements are immediate from the following more general result.

THEOREM 4.9. *β is the strong Mackey topology of the dual pair $(L^\infty(\mathcal{Q}), L^1(\mathcal{Q}))$. Even more is true: any $\sigma(L^1, L^\infty)$-countably compact subset of $L^1(\mathcal{Q})$ is β-equicontinuous.*

PROOF. A set $H \subset L^1(\mathcal{Q})$ is β-equicontinuous iff H is β_Q-equicontinuous for each $Q = \eta(a)\backslash\bigcup_{k=1}^\infty \eta(a_k)$. But if H is $\sigma(L^1, L^\infty)$-countably compact, then so is H countably compact as a subset of the dual (L^∞, β_Q). Since C_Q has a countable approximate identity it follows as in [26, 3.5(b)] from [5] that H is β_Q-equicontinuous.

This last result is really nothing more than the Conway-LeCam result for measures on σ-compact locally compact spaces, extended through the inductive limit to $L^1(\mathcal{Q})$. The result [26, 2.6] for the dual pairing (C_b, M_σ) has exactly the same origin and proof. All of these can be ultimately founded on Aleksandrov's theorem [30, Theorem 14] that the sequential limit of σ-additive functionals is σ-additive.

Theorem 4.8 in its present form is not very meaningful: we have not indicated what β_∞-convergence is on $L^\infty(X, \Sigma, \mu)$. This will be done in §8, (8.4), when we study Boolean measure algebras. For the moment we return to the general theory of $L^1(\mathcal{Q})$.

Because of 3.1(a), each $\hat{\mu} \in L^1(\mathcal{Q})$ is a bounded linear functional on $L^\infty(\mathcal{Q})$. We norm $L^1(\mathcal{Q})$ by the norm induced on $L^1(\mathcal{Q})$ as a subset of the Banach space dual of $L^\infty(\mathcal{Q})$. That is, let

$$\|\hat{\mu}\|_1 = \sup\{|\hat{\mu}(f)|: \|f\| \leqslant 1\}.$$

Now by 3.2, $L^1(\mathcal{Q})$ is closed in the Banach dual of $L^\infty(\mathcal{Q})$ and hence $(L^1(\mathcal{Q}), \|\ \|_1)$ is a Banach space. We return to this structure in §5.

Because of 4.9, we want to characterize the weak* compact subsets of $L^1(\mathcal{Q})$. This result will be one of the more important justifications for introducing β and L^1, and is the analogue of the (C_b, M)-duality theorem that weak* compact sets are uniformly tight, at least, in some cases (see [5, Theorem 2.2], [26, Theorem 5.1] and [28, 5.2] for further analogues).

For a fixed $\hat{\nu} \in L^1(\mathcal{Q})$ and $a \in \mathcal{Q}$, let $\hat{\nu}_a(f) = \hat{\nu}(\chi(a)f)$. From 3.3 and 4.3, $\hat{\nu}_a \in L^1(\mathcal{Q})$ since, if $b_n \nearrow b$ in $\mathcal{Q}$, then $ab_n \nearrow ab$. We give a direct proof of the following

Proposition 4.10. *If $a_n \nearrow a$ in $\mathcal{Q}$, then $\|\hat{\nu}_{a_n} - \hat{\nu}_a\|_1 \to 0$.*

Proof. Let $Q = \eta(a)\backslash\bigcup_{n=1}^\infty \eta(a_n)$ and $\varepsilon > 0$. Then $W = \{f: |\hat{\nu}(f)| < \varepsilon\}$ is a β-neighborhood of zero. Let $\xi \equiv 0$ on Q, continuous on S, such that $V_\xi \subset W$. Cover $\{\xi \geqslant \frac{1}{2}\}$ by $\eta(a_{n_0})$ and $S\backslash\eta(a)$ for some n_0 as in 4.1. Let $f \in L^\infty(\mathcal{Q})$, $\|f\| \leqslant 1$. Now for $n \geqslant n_0$, $|(\chi(a) - \chi(a_n))\xi \cdot f| \neq 0$ only if $x \in \eta(a)\backslash\eta(a_{n_0}) \subset \{\xi < \frac{1}{2}\}$. Since $\|f\| \leqslant 1$ and $\|\chi(a) - \chi(a_n)\| \leqslant 2$ then $\|[(\chi(a) - \chi(a_n))f]\xi\| \leqslant 1$ and

$$|(\hat{\nu}_a - \hat{\nu}_{a_n})(f)| = |\hat{\nu}(\chi(a) - \chi(a_n) \cdot f)| < \varepsilon$$

for $n \geqslant n_0$ and all $\|f\| \leqslant 1$. So, $\|\hat{\nu}_a - \hat{\nu}_{a_n}\|_1 \leqslant \varepsilon$ for $n \geqslant n_0$.

It is easy to see that 4.10 in fact characterizes those bounded functionals on $L^\infty(\mathcal{Q})$ which are β-continuous. The essence of equicontinuity is that a set of functionals behave like a single functional with respect to the defining continuity property. This suggests two of our more important theorems, 4.11 below and 7.6(2). The first of these, 4.11, says that β-equicontinuity is but the analogue of the conditions of the Dunford-Pettis theorem [10, p. 274] for ordinary L^1-spaces (see also Theorem 8.1 below) and the analogue of uniform tightness already observed (3.4).

THEOREM 4.11. *A subset H of $L^1(\mathcal{Q})$ is $\sigma(L^1, L^\infty)$-countably compact iff* (a) *H is $\| \|_1$-bounded and* (b) *if $a_n \nearrow a$ in $\mathcal{Q}$, then $\|\hat{\nu}_a - \hat{\nu}_{a_n}\|_1 \to 0$ uniformly over $\hat{\nu} \in H$.*

PROOF. By 4.9, a $\sigma(L^1, L^\infty)$-countably compact set is β-equicontinuous. Hence $H \subset W^0 \subset B_r^0$ (W^0 is the polar of W [22]) for some $r > 0$ and some $W \in \beta$. Hence (a) holds. For (b), if $a_n \nearrow a$, choose $\xi = 0$ on $Q = \eta(a)\backslash\bigcup_{n=1}^{\infty}\eta(a_n)$ so that $(1/\varepsilon)V_\xi \subset W$ where $V_\xi = \{f \in L^\infty(\mathcal{Q}): \|f\xi\| < 1\}$. Then because $(1/\varepsilon)H \subset V_\xi^0$, we have $|\hat{\nu}(f)| \leq \varepsilon$ for $\|\xi f\| < 1$ uniformly over $\hat{\nu} \in H$. Choosing n_0 as in 4.10, we obtain $\|\hat{\nu}_a - \hat{\nu}_{a_n}\|_1 < \varepsilon$ uniformly over $\hat{\nu} \in H$.

Conversely, under (a) and (b), the polar $W = H^0$ of H in $L^\infty(\mathcal{Q})$ is absorbent and absolutely convex. If $a_n \nearrow a$, pick n_0 such that $\|\hat{\nu}_a - \hat{\nu}_{a_n}\|_1 < 1/2r$ for $n \geq n_0$ where $r > 0$ is fixed. Let $\xi = 1 - [\chi(a) - \chi(a_{n_0})]$ and suppose $f \in B \cap (1/2\alpha)V_\xi$ where $\alpha = \sup\{\|\hat{\nu}\|_1 : \hat{\nu} \in H\}$. Then $\hat{\nu} \in H$ implies that

$$|\hat{\nu}(f)| \leq |\hat{\nu}((1 - [\chi(a) - \chi(a_{n_0})])f)| + |\hat{\nu}(f \cdot (\chi(a) - \chi(a_{n_0})))|$$

$$\leq \|\xi f\|\,\|\hat{\nu}\|_1 + \|\hat{\nu}_a - \hat{\nu}_{a_n}\|_1\,\|f\| \leq 1.$$

Hence, $H^0 \cap B_r \supset (1/2\alpha)V_\beta \cap B$ for each $r > 0$ and by 3.2 $W = H^0$ is β-equicontinuous. Thus $H \subset W^0$ is weak* relatively compact in $L^1(\mathcal{Q})$.

REMARK. Of course a proof of 4.11 can be drawn from 3.4. We felt it was worthwhile to give a direct proof to emphasize how a set of such functionals acts like only one and to see again how β works in $L^\infty(\mathcal{Q})$. We also note that 4.11(b) can be thought of as: $\{|\nu|: \hat{\nu} \in H\}$ is uniformly countably additive, where $|\nu|$ is the measure $|\nu|(a) = |\hat{\nu}| \circ \chi(a)$ of 4.4(b).

As an easy application we have

THEOREM 4.12. *Multiplication is β-separately continuous on $L^\infty(\mathcal{Q})$.*

PROOF. Let $H \subset L^1(\mathcal{Q})$ be β-equicontinuous and let $g \in L^\infty(\mathcal{Q})$. If $G = \{\hat{\nu}: \hat{\nu}(f) = \hat{\mu}(gf), \hat{\mu} \in H\}$, then for $a_n \nearrow a$ in $\mathcal{Q}$,

$$\|\hat{\nu}_{a_n} - \hat{\nu}_a\|_1 \leq \|g\|\,\|\hat{\mu}_{a_n} - \hat{\mu}_a\|_1$$

converges uniformly to 0 over $\hat{\nu} \in G$ by 4.11.

Hence, if $f_\alpha \to^\beta f$, then $\hat{\mu}(gf_\alpha - gf) \to 0$ uniformly over $\hat{\mu} \in H$. By [22, p. 48] $gf_\alpha \to^\beta gf$.

Note that on the sets B_r multiplication is jointly continuous as well.

5. **Topology and order in L^1 and L^∞.** A complete study of the natural order properties of $L^\infty(\mathcal{Q})$ is found in Fremlin [11]. Strict topologies lose some of

their unwieldy character as inductive limits when the interplay with the natural order structures of functions and measures is exploited; see [26, 5.5 and 6.3], where for C_b-spaces it is shown that β is the finest locally convex topology for which Dini's classical monotone-implies-uniform convergence theorem holds. We obtain an analogue for β on $L^\infty(\mathcal{A})$ herein (5.7). Our principal references for this section are [21] and [23].

We are trying to show the extent to which the dual pairings (C_b, M) and (L^∞, L^1) can be viewed in the same way and to exploit the identification of measurable functions with continuous functions on the Stone space. In this section we mildly exploit the identification of functionals $\hat\mu \in L^1(\mathcal{A})$ with bounded regular Borel measures μ^* on S.

Digressing for a moment we let $L(\mathcal{A})$ denote the $\bar\beta$-dual of $L^\infty(\mathcal{A})$. It is easy to see that versions of 3.1(a), 4.1, 4.3 and 4.6 hold in $L(\mathcal{A})$ with sequences replaced by nets and that $L(\mathcal{A}) \subset L^1(\mathcal{A})$. We will study this relation further in the next section but obtain a preliminary result below (5.4).

Define $f \geqslant g$ in $L^\infty(\mathcal{A})$ if $f(x) \geqslant g(x)$ for all $x \in S$ and $\hat\mu \geqslant \hat\nu$ in $L^1(\mathcal{A})$ if $\hat\mu(f) \geqslant \hat\nu(f)$ for all $f \in L^\infty_+ = \{f \in L^\infty : f \geqslant 0\}$. Let $L^1_+ = \{\hat\mu \in L^1(\mathcal{A}): \hat\mu \geqslant 0\}$ and $L_+ = \{\hat\mu \in L(\mathcal{A}): \hat\mu \geqslant 0\}$. The results of this section hold for L and $\bar\beta$ as they do for L^1 and β but we develop it in these latter terms.

Under the ordering above, $L^\infty(\mathcal{A})$ is a Banach lattice under the supremum norm and in fact an M-space [11, p. 24]. Its Banach dual $M(S)$ is itself a Banach lattice and an L-space [11, p. 26]. In fact, $M(S)$ is order complete [20, p. 24]. From 3.1(a), we may identify $L^1(\mathcal{A})$ with a subspace of $M(S)$ and the $M(S)$-norm is the $\| \ \|_1$ defined on $L^1(\mathcal{A})$ following 4.9. Most importantly, as an element of the ordered space $M(S)$, the positive part of the functional $\hat\mu \in L^1(\mathcal{A})$ is the functional $\hat\mu^+ \in L^1(\mathcal{A})$ of 2.2. From this it follows that if $\hat\nu \in L^1(\mathcal{A})$ and $\hat\mu \in M(S)$ such that $|\hat\mu| \leqslant |\hat\nu|$, then $\hat\mu \in L^1(\mathcal{A})$ (4.4(b)) and hence that $L^1(\mathcal{A})$ is both a solid subspace of $M(S)$ and a lattice ideal in $M(S)$ [20, p. 35]. Since $L^1(\mathcal{A})$ is complete in the $\| \ \|_1$ by 3.2, $L^1(\mathcal{A})$ is a Banach lattice and hence is itself an L-space. Now, any positive linear functional on $L^\infty(\mathcal{A})$ is $\| \ \|$-continuous hence $M(S)$ is order complete. If, moreover $K \subset L^1(\mathcal{A})$ is increasing and bounded above by some $\phi \in M(S)$, then $\hat\mu = \sup K \in M(S)$ is in fact in $L^1(\mathcal{A})$. For, we can suppose that elements of K are positive functionals and then, for $a_n \nearrow a$ in $\mathcal{A}$,

$$\sup_n \hat\mu(\chi(a_n)) = \sup_n \sup_{\hat\nu \in K} \hat\nu(\chi(a_n)) \geqslant \sup_{\hat\nu \in K} \sup_n \hat\nu(\chi(a_n))$$
$$= \sup_{\hat\nu \in K} \hat\nu(\chi(a)) = \hat\mu(\chi(a)) \geqslant \sup \hat\mu(\chi(a_n)).$$

We have proven

THEOREM 5.1. $(L^1(\mathcal{A}), \| \ \|_1)$ *is an order complete Banach lattice and a band in*

$M(S)$. Moreover, $L^1(\mathfrak{A}) = L^1_+ - L^1_+$ and for $\hat{\mu} \in L^1(\mathfrak{A})$, $\|\hat{\mu}\|_1 = \hat{\mu}^+(1) + \hat{\mu}^-(1)$.

The final property follows from the fact that $L^1(\mathfrak{A}) \subset M(S)$ and that 1 is the order unit for $L^\infty(\mathfrak{A})$. Though 5.1 suggests a number of questions concerning $L^1(\mathfrak{A})$, particularly with respect to Kakutani's L-space representation theorem and with respect to [11], we will make little use of 5.1 in what follows. Chiefly we need to generate $L^1(\mathfrak{A})$ from L^1_+, which is already guaranteed by 3.3.

The Riesz representation theorem gives another representation of $L^1(\mathfrak{A})$ distinct from that of 4.5. For we can identify $M(S)$ with the space of bounded regular Borel measures on S under the total variation norm. That is, $\hat{\mu} \in M(S)$ iff $\hat{\mu}(f) = \int_S f\,d\mu^*$ for some unique bounded regular Borel μ^* on S and $\|\hat{\mu}\|_1 = |\mu^*|(S)$. The correspondence $\hat{\mu} \to \mu^*$ is a Riesz space isomorphism [11, p. 14]; in particular, for the functionals $|\hat{\mu}|$ and $\hat{\mu}^+$ of 2.2, $|\hat{\mu}|^* = |\mu^*|$, the total variation of the measure μ^*, and, $(\hat{\mu}^+)^* = \mu^{*+}$, the positive part of the signed measure μ^*. We will use this notation in the remainder of the paper and so identify $L^1(\mathfrak{A})$ with a subspace of $M(S)$ itself henceforth identified with the space $M_t(S)$ (notation follows 2.1).

THEOREM 5.2.

$$L^1(\mathfrak{A}) = \{\mu^* \in M(S): |\mu^*|(Q) = 0 \quad \text{for all } Q = \eta(a)\setminus\textstyle\bigcup_{n=1}^{\infty}\eta(a_n) \text{ where } A_n \nearrow a \text{ in } \mathfrak{A}\}.$$

PROOF. Since $\hat{\mu} \in L^1(\mathfrak{A})$ puts $|\hat{\mu}| \in L^1(a)$, then $|\mu^*|(Q) = 0$ by 4.6. Conversely, if $\mu^* \in M(S)$ has this property for all such Q, and we define $\hat{\mu}(f) = \int_S f\,d\mu^*$ for $f \in L^\infty(\mathfrak{A})$, then $|\hat{\mu}|$ is a bounded linear functional on $L^\infty(\mathfrak{A})$ for which $\mu = |\hat{\mu}| \circ \chi$ is a measure on $\mathfrak{A}$. By 4.4, we are done.

The corresponding results 5.1 and 5.2 for $L(\mathfrak{A})$ are clear with Q replaced by $\eta(a)\setminus\bigcup_\alpha\eta(a_\alpha)$, $a_\alpha \nearrow a$ in $\mathfrak{A}$. We have not considered replacing Q by any closed nowhere dense subset of S though this obviously can be carried along in the same manner. More to the point of Boolean algebras, one can repeat all of our material thus far and much of what follows for sets $Q = \eta(a)\setminus\bigcup_{\alpha\in\Gamma}\eta(a_\alpha)$ where Γ is a set of some bounded cardinality m (for example, vis-a-vis m-complete Boolean algebras).

We turn now to what are for our purposes the principal results of this section.

THEOREM 5.3. *β is the topology of uniform convergence on the $\sigma(L^1, L^\infty)$-compact subsets of L^1_+.*

PROOF. Let τ be this latter topology. From 4.9, $\tau \leqslant \beta$. Conversely, from [22, pp. 12 and 79] β has a base of closed neighborhoods of zero consisting of sets of the form $W = \bar{V}$, where V is the absolutely convex hull of sets $\bigcup_Q V_{\xi_Q}$ where

$\xi_Q \in C_Q$ for each $Q = \eta(a)\backslash\bigcup_{n=1}^{\infty}\eta(a_n)$, $a_n \nearrow a$ in $\mathcal{A}$. Let $H = \{\hat{\mu}^+ : \hat{\mu} \in W^0\}$. Then (as in [26, 5.5]) $H/2 \subset V_{\xi_Q}^0$ for each Q so that H is β-equicontinuous. But then $W^0 \subset K - K$ where K is the $\sigma(L^1, L^\infty)$-closure of H. Since $W = W^{00}$ and K is $\sigma(L^1, L^\infty)$-compact, this makes W a τ-neighborhood, completing the proof.

The theorem above carries over to $\bar{\beta}$ and $L(\mathcal{A})$; its proof does not–the second sentence must be replaced by an argument like that found in [26, p. 323]. Using this and 5.3 we have

COROLLARY 5.4. $\beta = \bar{\beta}$ *iff* $L^1(\mathcal{A}) = L(\mathcal{A})$.

We will have more to say on the equality $L = L^1$ in a later section.

Now we consider $(L^\infty(\mathcal{A}), \beta)$ as an ordered topological vector space.

THEOREM 5.5. *L_+^∞ is a normal cone for (L^∞, β) and the lattice operations $f \to f^+$, $f \to |f|$, $(f, g) \to (f \vee g)$ are all β-continuous. Equivalently, β is locally solid.*

PROOF. The equivalence of the two statements is due to Schaefer [23, p. 234]. That the first is true follows exactly as in [26, 6.1].

COROLLARY 5.6. *A β-equicontinuous subset of $L^1(\mathcal{A})$ is a subset of some difference $H - H$ where $H \subset L_+^1$ is β-equicontinuous. This follows from* 5.5 *and* [23, *p.* 66].

THEOREM 5.7. *Let $\mathcal{A}$ be σ-complete. If $\{f_n\} \subset L^\infty(\mathcal{A})$ is a decreasing sequence with $\wedge f_n = 0$, then $f_n \to^{\beta} 0$. Moreover, β is the finest locally convex topology on $L^\infty(\mathcal{A})$ for which the lattice infimum (supremum) in $L^\infty(\mathcal{A})$ of a decreasing (increasing) sequence in $L^\infty(\mathcal{A})$ is the topological limit.*

PROOF. Suppose $f_n \geqslant f_{n+1}$ in $L^\infty(\mathcal{A})$, with infimum zero in $L^\infty(\mathcal{A})$ and pointwise infimum $f(x) = \inf_n f_n(x)$. Fix $\varepsilon > 0$.

Let $A = \{x \in S : f(x) \geqslant \varepsilon > 0\}$. We claim that $A \subset \eta(a)\backslash\bigcup_{k=1}^{\infty}\eta(a_k)$ where $a_k \nearrow a$ in $\mathcal{A}$. Let $A_{nk} = \{x : f_n(x) > \varepsilon - 1/k\}$. Now $A \subset \bigcap_{n,k} A_{nk}$ and hence the closed sets A (since f is upper-semicontinuous) and $S\backslash A_{nk}$ are disjoint for all n,k. Because S is compact we can find an $a_{nk} \in \mathcal{A}$ such that $A \cap \eta(a_{nk}) = \square$ and $\eta(a_{nk}) \supset S\backslash A_{nk}$. Let $a = \sup_{n,k} a_{nk}$ in $\mathcal{A}$. Now, $\eta(a) = S$ for, $\eta(e) = S$, and if $b = e - a \neq 0$, and $x \in \eta(b)$, then $x \notin \eta(a_{nk}) \supset S\backslash A_{nk}$ for all n, k, whence $f(x) \geqslant \varepsilon$. That is, we would then have $\eta(b) \subset A$ and from this a continuous function, $\varepsilon\chi(b)$, larger than zero but smaller than every f_n. Hence $a = e$, $S = \eta(a)$ and

$$A \subset \bigcap_{n,k} S\Big\backslash\eta(a_{nk}) = \eta(a)\Big\backslash\bigcup_{n,k}\eta(a_{nk}).$$

Letting $a_n = \max_{i,j \leqslant n} a_{ij}$, we have $a_n \nearrow a$, and our initial claim is met.

But now, according to 5.2, $\mu^*(A) = 0$ for the Borel measure μ^* representing a fixed $\hat{\mu} \in L^1_+$. Hence, $\int_S f\,d\mu^* = 0$. But $\hat{\mu}(f_n) = \int f_n\,d\mu^* \searrow \int_S f\,d\mu^*$, so that for any $\hat{\mu} \in L^1_+$, $\hat{\mu}(f_n) \to 0$. Now, if $H \subset L^1_+$ is weak* compact, it follows from Dini's classical convergence theorem that $\hat{\mu}(f_n) \to 0$ uniformly over $\hat{\mu} \in H$, whence $f_n \to^\beta 0$.

To finish the proof let $\mathfrak{U} = \{W \subset L^\infty(a)$: W is absolutely convex, absorbent and, for all decreasing sequences $\{f_n\} \subset L^\infty(\mathfrak{a})$ with $\wedge f_n = 0$, one has $f_n \in W$ eventually$\}$. Then [22, p. 10] $\mathfrak{U}$ is a base for a locally convex topology $\mathfrak{T}$ on $L^\infty(\mathfrak{a})$. From the above $\beta \leqslant \mathfrak{T}$ and from [26, 6.3], $\mathfrak{T} \leqslant \beta_1 = \|\ \|_l$. Hence any $\mathfrak{T}$-continuous functional ϕ is $\|\ \|$-bounded. If $a_n \searrow 0$ in $\mathfrak{a}$ then $\{\chi(a_n)\}$ is decreasing with $\wedge \chi(a_n) = 0$. Hence $\phi \circ \chi$ meets 4.4(c) and ϕ is β-continuous. By 4.9, $\mathfrak{T} \leqslant \beta$ and we are done.

Remark. It is possible to give an order theoretic proof of $\mathfrak{T} \leqslant \beta$ by arguing that ϕ^+ is $\mathfrak{T}$-continuous and using 4.3. Since a $\bar{\beta}$ analogue of 4.3 holds with nets replacing sequences it then becomes possible to obtain 5.7 for decreasing nets and the topology $\bar{\beta}$, assuming that $\mathfrak{a}$ is complete. The initial part of the argument goes through exactly and the second part can be based essentially on a repeat of an argument in [12] that if $W \in \mathfrak{T}$, then $W^{0+} \in \beta_Q$ for each Q (but see also 8.4). Given then that for a decreasing net $\{f_\alpha\}$ with $\wedge f_\alpha = 0$ one has $f_\alpha \to^{\bar{\beta}} 0$, it then follows by extending 4.3 to nets $a_\alpha \searrow 0$ that the $\bar{\beta}$-dual $L(\mathfrak{a})$ is Fremlin's $L^\#(\mathfrak{a})$ of real linear functionals on $L^\infty(\mathfrak{a})$ expressible as the difference of positive order continuous functionals [11, 16 and 44].

Proposition 5.8. *The topology β is the finest locally convex topology coarser than $\|\ \|$ on $L^\infty(\mathfrak{a})$ for which χ: $\mathfrak{a} \to L^\infty(\mathfrak{a})$ is a vector measure.*

For, arguing as above, a locally convex topology for which χ is a measure and for which linear functionals on $L^\infty(\mathfrak{a})$ are bounded yields the dual $L^1(\mathfrak{a})$ by 4.3 and hence is coarser than β.

Proposition 5.9. *If β is a Hausdorff topology on $L^\infty(\mathfrak{a})$ and $\{f_n\}$ is an increasing sequence in $L^\infty(\mathfrak{a})$ and $f_n \to^\beta f$, then $f = \vee f_n$.*

Proof. First, $f_n \leqslant f$ for all n. For if not, then for $k \geqslant n$ and some $a \in \mathfrak{a}$ and $\delta > 0$, $f(x) + \delta < f_k(x)$ for all $x \in \eta(a)$. Since β is Hausdorff there is a $\hat{\mu} \in L^1_+$ such $\hat{\mu}(\chi(a)) > 0$. Now $\hat{\mu}_a(f_n) \to \hat{\mu}_a(f)$ since $\hat{\mu}_a \in L^1(\mathfrak{a})$, but yet, $\hat{\mu}_a(f_k) = \hat{\mu}(\chi(a) f_k) \geqslant \hat{\mu}((f+\delta)\chi(a)) \geqslant \hat{\mu}_a(f) + \delta\hat{\mu}(\chi(a))$, a contradiction. Similarly one can prove that if $f_n \leqslant g \in C(S)$, then $g \leqslant f$, completing the proof.

6. **L^∞ as the dual of L^1.** In this paper we of course have turned things around to define L^1 as the dual of L^∞, under the topology β, the reverse pairing being more familiar. In this section we study the extent to which $L^\infty(\mathfrak{a})$ is the norm dual of $L^1(\mathfrak{a})$.

There are two distinct ways in which to organize this and the remaining sections. We can begin with the study of Boolean algebras which admit a normalized strictly positive measure μ (for the definition see 6.5 below) and proceed to generality by way of a form of the Radon-Nikodým theorem (8.13(b)), or begin in generality and show how matters ultimately and naturally reduce to this important special case. It is believed that by beginning in generality we can more clearly see how the pieces fit and the same time relate and make use of what is already known.

In asking that $L^\infty(\mathcal{A}) = C(S)$ be the dual of the Banach lattice $L^1(\mathcal{A})$ we are immediately guided by Dixmier's result [7], that $C(S)$ can be such a dual iff S is hyperstonian, that is, iff $\mathcal{A}$ is a complete Boolean algebra and the normal measures on S separate points of $C(S)$ [7], and, in this case, $L^\infty(\mathcal{A}) = C(S)$ is the dual of these normal measures. Beginning in generality then, we ask when is $L^1(\mathcal{A})$ the class of normal measures on S? First some technical digressions.

Even when $\mathcal{A}$ is complete, the dual of $L^1(\mathcal{A})$ under the norm $\| \|_1$ defined in 4.9 need not be $L^\infty(\mathcal{A})$. Consider the example following 4.5, where $L^1(\mathcal{A}) = \{0\}$. To say that $L^\infty(\mathcal{A}) = (L^1(\mathcal{A}), \| \|_1)'$ we mean that the map ψ: $L^\infty(\mathcal{A}) \to (L^1(\mathcal{A}), \| \|_1)'$ by $\psi(f)(\hat{\nu}) = \hat{\nu}(f)$ is 1-1 and onto. In particular ψ is 1-1 iff L^1 separates points of L^∞–i.e., β is Hausdorff. This condition not only makes ψ 1-1 but an isometry as well, and the norms $\| \|$ and $\| \|_1$ are dual norms (see also [11, 52G]).

PROPOSITION 6.1. *If β is Hausdorff, then* $\|f\| = \sup\{|\hat{\mu}(f)|: \|\hat{\mu}\|_1 \leqslant 1, \hat{\mu} \in L^1(\mathcal{A})\}$.

PROOF. Let $\|f\|'$ denote the norm on the right. Clearly, $\|f\|' \leqslant \|f\|$. Suppose $f = \sum_{i=1}^n \alpha_i \chi(a_i) \in \mathbb{S}(\mathcal{A})$ with $a_i a_j = 0$ for $i \neq j$. Then $\|f\| = |\alpha_i|$ for some i. Let $a = a_i$ and, since β is Hausdorff, choose $\hat{\mu} \in L^1_+$ such that $\mu(a) > 0$. Let $\hat{\nu}(f) = \hat{\mu}(\chi(a)f)/\mu(a)$. Then $\hat{\nu} \in L^1_+$ (3.3 and 4.3), $\|\hat{\nu}\|_1 \leqslant 1$, and

$$|\hat{\nu}(f)| = |\alpha_i|\hat{\nu}(\chi(a)) = \|f\|.$$

Thus the two norms agree on the $\| \|$-dense subset $\mathbb{S}(\mathcal{A})$, and since $\| \|' \leqslant \| \|$, agree on all of $L^\infty(\mathcal{A})$.

REMARK. Under the hypothesis of 6.1, it follows from the uniform boundedness principle that the β bounded sets in $L^\infty(\mathcal{A})$ are $\| \|$-bounded just as for C_b-spaces [26, 4.7].

We organize the remainder of this section by taking Dixmier's results [7] (or [20, p. 145]) as given and relate our space $L^1(\mathcal{A})$ to his study.

Dixmier [7] calls a measure $\mu^* \in M(S)^+$ normal if $\mu^*(f) = \sup \mu^*(f_\alpha)$ (where $\mu^*(f) \equiv \int_S f d\mu^*$) for any increasing net $\{f_\alpha\} \in C(S)$ with (lattice) supremum $f \in C(S)$. We denote the vector subspace of $M(S)$ of difference of positive normal measures by $N(S)$, and recalling our identification of $L^1(\mathcal{A})$

with a subspace of $M(S)$ (§5), note the readily apparent

PROPOSITION 6.2. $N(S) \subset L^1(\mathcal{Q})$.

For, $\chi(a) = \bigvee_n \chi(a_n)$ when $a_n \nearrow a$ in $\mathcal{Q}$.

A cardinal Γ is of measure zero if any finite valued countably additive measure on 2^Γ is a countable sum of point measures; i.e., $\mu(\Gamma) = \mu(\Gamma_0)$ for some countable $\Gamma_0 \subset \Gamma$. Both $\aleph_0$ and $\aleph_1$ have measure zero, and so does c if one assumes the continuum hypothesis. Further discussion can be found, for example, in [14]. The point we must make is that if there is a cardinal Γ not of measure zero with nonzero measure m such that $m(\{\gamma\}) = 0$ for each $\gamma \in \Gamma$, then m defines an element $\hat{m} \in L^1(2^\Gamma)$ which (by 4.3) cannot be normal, since $1 = \sup\{\chi(a): a \in 2^\Gamma$ is a finite subset of $\Gamma\}$. Hence in a discussion of $L^\infty(\mathcal{Q})$ as the dual of $L^1(\mathcal{Q})$ the issue of cardinals of measure zero is unavoidable.

We avoid this difficulty initially by dealing with the analogue of σ-finite measure spaces.

THEOREM 6.3. *If $\mathcal{Q}$ is σ-complete and satisfies the countable chain condition, then $L^1(\mathcal{Q}) = N(S)$. Moreover, β is Hausdorff iff $(L^1(\mathcal{Q}), \| \|_1)' = L^\infty(\mathcal{Q})$.*

PROOF. Let $f_0(x) = \sup f_\alpha(x)$ be the pointwise supremum of the increasing family $\{f_\alpha\}$, with $f = \bigvee_\alpha f_\alpha$. Then $f \geqslant f_0$ and $A = \{x: f(x) > f_0(x)\} = \bigcup_{n=1}^\infty \{x: f(x) - f_0(x) \geqslant 1/n\} = \bigcup_{n=1}^\infty A_n$ is a countable union of closed and nowhere dense subsets of S. For, if there is an $a \in \mathcal{Q}$ with $\eta(a) \subset A_n$ for some n, then the continuous function $f - \chi(a)/n \geqslant f_0 \geqslant f_\alpha$ for all α, a contradiction. Because f_0 is lower-semicontinuous. A_n is closed.

Now let $B = \{a \in \mathcal{Q}: \eta(a) \subset S \backslash A_n\}$ for fixed n. By [16, p. 62], because $\mathcal{Q}$ is σ-complete with c.c.c., B has a supremum $a \in \mathcal{Q}$. Again, because $\mathcal{Q}$ has c.c.c., there must be a sequence $a_k \in B$ such that $a_k \nearrow a$ in $\mathcal{Q}$. We claim that $A_n \subset Q = \eta(a) \backslash \bigcup_{k=1}^\infty \eta(a_k)$. For suppose there is an $x \in A_n \backslash Q$. Each $\eta(a_k)$ misses A_n by choice of a_k, and thus $x \notin \eta(a)$ as well. Now, there is some $b \in \mathcal{Q}$ with $x \in \eta(b)$ and $\eta(b) \cap \eta(a) = \square$. Now A_n being closed and nowhere dense implies that $\eta(b) \cap S \backslash A_n \neq \square$. Hence $\eta(b) \cap \eta(c) \neq \square$ for some $c \in B$. But then since $\eta(a) \supset \eta(c)$, $\eta(b) \cap \eta(a) = \square$ and we have a contradiction.

Now, for any $\hat{\mu} \in L^1_+$, $\mu^*(A_n) \leqslant \mu^*(Q) = 0$ by 4.6. Hence, $\mu^*(A) = 0$, so that $\mu^*(f) = \mu^*(f_0) = \sup \mu^*(f_\alpha)$ and μ^* is normal.

Finally, under these hypotheses $\mathcal{Q}$ is complete [16], hence S is extremally disconnected. If β is Hausdorff, $N(S) = L^1(\mathcal{Q})$ separates points of $L^\infty(\mathcal{Q})$, and by Dixmier's result, $L^\infty(\mathcal{Q}) = (L^1(\mathcal{Q}), \| \|_1)'$.

In 8.13 we indicate how a direct proof of 6.3 can be made. To continue, we note that, if, on the other hand, c is a cardinal of measure zero, then

$$L^1(\mathcal{Q}) = l^1[0,1] = \left\{\{a_t\}: t \in [0,1], \sum |a_t| < \infty\right\}$$

where $\mathcal{Q} = 2^{[0,1]}$. Hence, $L^1(\mathcal{Q})' = L^\infty(\mathcal{Q}) = l^\infty[0,1]$, but $\mathcal{Q}$ does not meet c.c.c.

At this point the topology $\bar{\beta}$ and its dual $L(\mathcal{Q})$ become useful in a natural way.

THEOREM 6.4. $L(\mathcal{Q}) = N(S)$. *If* $\mathcal{Q}$ *is complete, then* $\bar{\beta}$ *is Hausdorff iff* $(L(\mathcal{Q}), \| \|_1)' = L^\infty(\mathcal{Q})$.

The proof of 6.4 is just like that of 6.3, only easier. Define f_0 and A_n as in 6.3 and show that if $\hat{\mu} \in L(\mathcal{Q})$, then again $\mu^*(A_n) = 0$ just as above. From this, μ^* is normal and, conversely, a normal μ^* defines $\hat{\mu}(f) = \mu^*(f)$ in $L(\mathcal{Q})$.

In general, then $N(S) = L(\mathcal{Q}) \subset L^1(\mathcal{Q})$. The remainder of this discussion takes its ideas from Dixmier's result that a hyperstonian S is "nearly" the union of supports of normal measures on S [7, Lemma 2].

If $c \in \mathcal{Q}$, then $c\mathcal{Q} = \{a \in \mathcal{Q}: a \leqslant c\}$ is a Boolean ideal in $\mathcal{Q}$ which is complete if $\mathcal{Q}$ is. A normalized strictly positive measure on $c\mathcal{Q}$ is a measure μ on $c\mathcal{Q}$ such that $\mu(c) = 1$ and $\mu(a) > 0$ for all $a \in c\mathcal{Q}$, $a \neq 0$.

LEMMA 6.5. *There is a subset* B *of* $\mathcal{Q}$ *consisting of pairwise disjoint elements* $c_\alpha \in \mathcal{Q}$ *such that each* $c_\alpha\mathcal{Q}$ *admits a normalized strictly positive measure* μ_α *and,* B *is maximal with respect to these two properties. If* $\mathcal{Q}$ *is complete, then* $\bar{\beta}$ *is Hausdorff iff* $\bigvee B = \bigvee_B b = e$.

PROOF. Let $D = \{(c,\mu): \mu$ is a strictly positive normalized measure on $c\mathcal{Q}\}$ and let $\mathfrak{D}$ consist of all subsets D' of D such that $(c',\mu'), (c'',\mu'') \in D' \in \mathfrak{D}$ and $c' \neq c''$ implies $c'c'' = 0$. Order $\mathfrak{D}$ by set inclusion. Then every increasing chain in $\mathfrak{D}$ is bounded above. Let D_0 be a maximal element in $\mathfrak{D}$ and let $B = \{b: (b,\mu) \in D_0\}$, to complete the first part of the argument.

Now suppose that $\mathcal{Q}$ is complete. If $\bar{\beta}$ is Hausdorff and (in particular if B were empty and $\bigvee B = 0$), there is a $\bar{\beta}$-continuous $\hat{\mu}$, and hence normal μ^*, such that $\mu^*(\eta(e - \bigvee B)) > 0$. Let ν^* be the restriction of μ^* to $\eta(e - \bigvee B)$. Clearly ν^* is normal. Let S_0 be the support of ν^*. Then S_0 is clopen [7, Proposition 3] so there is a $c \in \mathcal{Q}$ with $\eta(c) = S_0$. If $a \leqslant c$, then $\eta(a) \subset \eta(c)$ and $\nu^*(\eta(a)) > 0$. Let $\hat{\nu}$ be the $\bar{\beta}$-continuous functional corresponding to ν^* and $\nu = \hat{\nu} \circ \chi$ the corresponding measure on $\mathcal{Q}$. The set $D_0 \cup \{(c,\nu)\}$ contradicts the maximality of D_0.

Conversely, if $\bigvee B = e$ and $a \in \mathcal{Q}$, then $\bigvee_B ab = a$, and if $a \neq 0$, then $ab \neq 0$ for some b. If $(b,\mu) \in D_0$, then $\mu(ab) > 0$ so that $\mu(a) > 0$. This implies that $L(\mathcal{Q})$ separates points of $\chi(\mathcal{Q})$ and then that $\bar{\beta}$ is Hausdorff, completing the proof.

To continue, the subset $\mathcal{C} = c\mathcal{Q}$ is also a Boolean algebra with identity c. If $(c,\mu) \in D_0$, then since $\mu(a) > 0$ for all $a \in \mathcal{C}$, it is easy to see that $\mathcal{C}$ has c.c.c.

[16, p. 67] and is σ-complete if $\mathfrak{A}$ is. Moreover, $\hat{\mu} \in L^1(\mathcal{C})$ and by itself separates points of $L^\infty(\mathcal{C})$. Hence by 6.3, if $\mathfrak{A}$ is σ-complete $L^1(\mathcal{C}) = L(\mathcal{C})$ and $L^1(\mathcal{C})' = L^\infty(\mathcal{C})$. Now, $L^\infty(\mathfrak{A})$ is the norm dual of $L(\mathfrak{A})$ iff $\mathfrak{A}$ is complete and $\bar{\beta}$ is Hausdorff–that is, iff there is a maximal D_0 for which $\vee B = e$, or, equivalently, (because B is pairwise disjoint) $\sum_{b\in B} b = e$ (in the natural sense). This last means that e is the Boolean sum of B in $\mathfrak{A}$. Rather than repeat any further Dixmier's work in this setting, we observe that essentially $L^\infty(\mathfrak{A})$ is the norm dual of $L^l(\mathfrak{A})$ iff $L^l(\mathfrak{A})$ is the "sum" of the spaces $L^1(\mathcal{C})$ for $c \in B$. The corresponding results for $L^1(\mathfrak{A})$, and whether or not $L^1(\mathfrak{A})$ is $L(\mathfrak{A})$, depends entirely on the "sum" of B over countably additive measures on $\mathfrak{A}$.

THEOREM 6.6. *If $\bar{\beta}$ is Hausdorff, $L^1 \neq L$ iff there is a measure ν on $\mathfrak{A}$ such that $\nu(e) = 1$ and $\nu(b) = 0$ for all $b \in B$. That is, $L^1 = L$ iff $\nu(e) = \sum_B \nu(b)$ for every positive measure on $\mathfrak{A}$.*

PROOF. For such a measure ν, the corresponding functional $\hat{\nu}$ cannot be normal since, with $\bar{\beta}$ Hausdorff, $\chi(e) = \vee \chi(b)$. Conversely, if $\hat{\mu} \in L^1 \backslash L$, then $\infty > \hat{\mu}(1) = \mu(e) \geqslant \sup\{\mu(\vee_{i=1}^n b_i): b_1, \ldots, b_n \in B\}$. Hence, $\mu(b) \neq 0$ for at most a countable subset $A \subset B$. Let $\nu(a) = \mu(a) - \sum_{b\in A} \mu(ba)$ for $a \in \mathfrak{A}$. Then $0 < \nu(e) < \infty$ and for $b \in B$, $\nu(b) = \mu(b) - \mu(b) = 0$. The measure $\nu/\nu(e)$ satisfies the theorem.

To continue, let $\mathfrak{B}$ denote the smallest complete subalgebra of $\mathfrak{A}$ containing B. Since B is pairwise disjoint with $\vee B = e$, $\mathfrak{B}$ is Boolean isomorphic to the Boolean algebra 2^B (under $\cap$ and $\triangle$), through the mapping $\sigma(E) = \vee_{b\in E} b$ for $E \subset B$. Let $l^1(B) = \{\{x_t\}: t \in B, -\infty < x_t < \infty \text{ and } \sum_B |x_t| < \infty\}$.

THEOREM 6.7. *If $\mathfrak{A}$ is complete and $\bar{\beta}$ is Hausdorff, then the following are equivalent*:

(1) $L^1(\mathfrak{A})' = L^\infty(\mathfrak{A})$.

(2) $\mu(e) = \sum_B \mu(b)$ *for all positive measures μ on $\mathfrak{A}$.*

(3) $L^1(\mathfrak{B})' = L^\infty(\mathfrak{B})$.

(4) *The cardinal of B is a cardinal of measure zero.*

(5) $L^1(2^B) = l^1(B)$.

(6) $L^1(2^B)' = L^\infty(2^B)$.

PROOF. (1) $\to$ (2) Let μ be a measure on $\mathfrak{A}$. Since $\mathfrak{A}$ is complete and $\bar{\beta}$ is Hausdorff, if $L^1(\mathfrak{A})' = L^\infty(\mathfrak{A})$, then $L^1(\mathfrak{A}) = L(\mathfrak{A})$ since $L(\mathfrak{A})$ is $\| \|_1$-closed in $L^1(\mathfrak{A})$. Hence, μ^* is normal and (2) follows.

(2) $\to$ (1) This is 6.6.

(1) $\to$ (4) If m is a measure defined on 2^B with $m(\{b\}) = 0$ for all $b \in B$, let $(b, \mu_b) \in D_0$ with $\mu_b(b) = 1$ and set $\mu(a) = \int_B \mu_b(a) m(db)$. (Here B is simply thought of as a subset of $\mathfrak{A}$.) Then μ is a measure on $\mathfrak{A}$, $\mu(e) = m(B) \neq 0$ yet $\mu(b) = \mu_b(b) m(\{b\}) = 0$ contradicting (2).

(4) $\to$ (5) The Stone space of 2^B is the Stone-Čech compactification of the

discrete topological space B [29]. Clearly $l^1(B) = M_t(B)$, the tight measures on B, while $L^1(2^B) = M_\sigma(B)$, the space of all countably additive measures defined on B. By (4) and Katetov's Theorem [30, Theorem 28] any measure on 2^B is tight, so $M_\sigma(B) = M_t(B)$ and (5) holds.

(5) $\to$ (6) Clearly $l^1(B)' = l^\infty(B)$, the space of bounded functions defined on the set B, and $l^\infty(B) = L^\infty(2^B)$.

(6) $\to$ (3) For, $\mathfrak{B}$ is Boolean isomorphic to 2^B under the mapping σ above.

(3) $\to$ (1) Since by (3), $l^1(B)' = l^\infty(B)$, then $l^1(B) = M_t(B)$ and the cardinal of B is of measure zero. If (1) fails then $L^1 \neq L$. For the measure ν of 6.6 above, define $m(E) = \nu(\bigvee_{b\in E} b)$ for $E \subset B$. Then $m\{b\} = 0$ for each $b \in B$ and $m(B) = \nu(e) > 0$, a contradiction.

It is clear now what, for complete Boolean algebras, the proper analogues are with (C_b, M) dual pairs. $L^1(\mathfrak{A})$ plays the role of $M_\sigma(X)$, $L(\mathfrak{A})$ the role of $M_\tau(X)$; β is analogous to β_1 and $\bar{\beta}$ to β in [26]. The result above is reminiscent of Katetov's result on the equality $M_\sigma = M_\tau$, with completeness of $\mathfrak{A}$ playing the role of paracompactness (e.g. see [28]).

The following result can be used to study β-completeness (8.15). We include it now although its proof depends on the later but independent result 7.6(a).

Theorem 6.8. *If $L^1(\mathfrak{A})' = L^\infty(\mathfrak{A})$ and $\{f_b : b \in B\} \subset B_r$ such that $\chi(b) f_b = f_b$ for each $b \in B$, then there is an $f \in B_r$ such that $f =^\beta \sum_B f_b$.*

Proof. By 5.3 it suffices to find an $f \in L^\infty(\mathfrak{A})$, $\|f\| \leqslant r$ such that $\lim_{B'} \langle f - \sum_{B'} f_b, \hat{\mu}\rangle = 0$ uniformly over any β-equicontinuous set $H \subset L^1_+$, where B' ranges over the finite subsets of B.

For $\hat{\mu} \in L^1(\mathfrak{A})$ set $\phi(\hat{\mu}) = \sum_B \langle f_b, \hat{\mu}\rangle$. Since for any finite subset $B' \subset B$ one has

$$\left|\sum_{B'} \langle f_b, \hat{\mu}\rangle\right| \leqslant \sum_{B'} \langle |f_b|, |\hat{\mu}|\rangle$$

$$\leqslant r \sum_{B'} \langle \chi(b), |\hat{\mu}|\rangle \leqslant r \sum_{B'} |\mu|(b) \leqslant r|\mu|(e)$$

the functional ϕ is both defined and bounded on $L^1(\mathfrak{A})$. By hypothesis there is a unique $f \in L^\infty(\mathfrak{A})$ such that $\langle f, \hat{\mu}\rangle = \phi(\hat{\mu}) = \sum_B \langle f_b, \hat{\mu}\rangle$ for all $\hat{\mu} \in L^1(\mathfrak{A})$.

Now, looking ahead, by 7.6(a), if H is β-equicontinuous in L^1_+, $\bigcup_{\hat{\mu}\in H} \bigcup_{n=1}^\infty \{b \in B : \mu(b) \geqslant 1/n\}$ is countable and by 6.7 and 4.11 we can suppose that there is an N such that $\mu(e - \sum_{i=1}^N b_i) < \varepsilon/2r$ for all $\hat{\mu} \in H$.

It then follows that if $B' \supset \{b_i : i \leqslant N\}$, then

$$\left|\left\langle f - \sum_{B'} f_b, \hat{\mu}\right\rangle\right| = \left|\sum_B \langle f_b, \hat{\mu}\rangle - \sum_{B'} \langle f_b, \hat{\mu}\rangle\right|$$

$$= \left|\sum_{B\setminus B'} \langle f_b, \hat{\mu}\rangle\right| \leqslant \sum_{B\setminus B'} r\langle \chi(b), \hat{\mu}\rangle \leqslant r\mu\left(e - \sum_1^N b_i\right) < \varepsilon$$

for all $\hat{\mu} \in H$ at once completing the proof.

Notice that the conclusion of the theorem implies 6.7(2), and hence the hypothesis $L^1(\mathfrak{A})' = L^\infty(\mathfrak{A})$, so that the converse does hold.

We close this section with an important result which indicates how strong is the hypothesis that β be Hausdorff. In [13], Gaifman gives an example of an algebra with c.c.c. that admits no strictly positive measure.

THEOREM 6.9. *Let $\mathfrak{A}$ be σ-complete with c.c.c. These are equivalent*:

(a) *β is Hausdorff.*

(b) $L^1(\mathfrak{A})' = L^\infty(\mathfrak{A})$.

(c) *$\mathfrak{A}$ admits a strictly positive normalized measure.*

PROOF. (a) → (b) is 6.3. We have already noted that (c) → (a). Given (a) we have by 6.3 and 6.4 that $L^1(\mathfrak{A}) = N(S) = L(\mathfrak{A})$. Hence $\bar{\beta}$ is Hausdorff, and for the set B of 6.5, $\bigvee B = e$. But B is pairwise disjoint and then countable by c.c.c. Each element $b_n \in B$ then carries a measure μ_n which is strictly positive on $\mathfrak{B}_n = \{a \in \mathfrak{A}: a \leqslant b_n\}$. Letting $\mu = \sum_{n=1}^{\infty} \mu_n/2^n$ we have for any $a \neq 0$, $a = \bigvee_n ab_n$, since $B = \{b_n\}$ and hence some $ab_n \neq 0$. But $\mu(a) \geqslant \mu_n(a)/2^n = \mu_n(b_n a)/2^n > 0$, hence μ is strictly positive.

Finally, paraphrasing [14] it has been shown by D. Scott [14, p. 8] that in Zermelo-Fraenkel set theory with Gödel's constructibility axiom there are no cardinals not of measure zero, and hence under these axioms, with $\mathfrak{A}$ complete and enough normal measures to distinguish elements of $\mathfrak{A}$, $L^1(\mathfrak{A}) = L(\mathfrak{A})$ and $\beta = \bar{\beta}$ (5.4). For algebras which are not assumed complete (e.g., σ-algebras of sets) β and $L^1(\mathfrak{A})$ appear to be of principal interest.

7. A homomorphism theorem and the equivalence of weak and weak* convergence in $L^1(\mathfrak{A})$. In this section we will show that even though, in general, $L^1(\mathfrak{A})' \supsetneqq L^\infty(\mathfrak{A})$, the weak topology $\sigma(L^1, L^{1\prime})$, and the weak* topology $\sigma(L^1, L^\infty)$, have the same compact sets. This result has its predecessors, among them Grothendieck's weak compactness theorem in $M(S)$ [10], where the Borel sets determine the weak topology on weakly compact subsets of $M(S)$; similar results are found in Fremlin [11, 83E].

There is a standard measure theoretic/functional analytic method for studying separable subsets, or converging sequences of measures $\{\bar{\mu}_n\}$, or weak compactness in $M(S)$ through Eberlein's weak compactness theorem. One sets $\bar{\mu} = \sum |\bar{\mu}_n|/2^n$ and obtains $\{\bar{\mu}_n\}$ as a subset of the range of a linear mapping T on $L^1(\bar{\mu})$ because each $\bar{\mu}_n$ is absolutely continuous with respect to $\bar{\mu}$ (e.g. see [10, p. 281]).

The analogue of this method in our setting is quite apparent and yields the major results in a simple manner based on 6.3. At the same time we see some advantages in having a "measure free" construction of $L^1(\mathfrak{A})$ to deal with initially. In the end we see in a very clear way just how close a β-

equicontinuous set of functionals not only behaves like, but is "almost", a single functional (7.9).

Throughout this section $\mathcal{A}$ is a σ-complete Boolean algebra. We are interested in homomorphic images $\mathcal{B} = \mathcal{A}/\mathcal{M}$ of $\mathcal{A}$ where $\mathcal{M}$ is a σ-ideal in $\mathcal{A}$ (that is, $\mathcal{M}$ is closed under countable suprema). Let τ be the natural homomorphism, $\tau: \mathcal{A} \to \mathcal{B}$, onto $\mathcal{B}$ with kernel $\mathcal{M}$. With $\mathcal{M}$ a σ-ideal, it follows [16, p. 56] that $\tau(\vee a_n) = \vee \tau(a_n)$ and that $\mathcal{B}$ is σ-complete as well. An example of such an $\mathcal{M}$ is of course $\mathcal{M} = \bigcap_{n=1}^{\infty} \mu_n^{-1}(0)$ where μ_n is a sequence of measures defined on some σ-algebra $\mathcal{A}$ with $\cap$, $\triangle$ the Boolean operations.

Let $S_{\mathcal{A}}$, $S_{\mathcal{B}}$ denote the respective Stone spaces of $\mathcal{A}$ and $\mathcal{B}$ and define $\rho: S_{\mathcal{B}} \to S_{\mathcal{A}}$ by $\rho(S)(a) = s(\tau a)$ where elements of the Stone space are viewed as 0-1 valued homomorphisms. The topologies being pointwise convergence on $\mathcal{B}$ and $\mathcal{A}$ respectively, it follows that ρ is a homeomorphism and that we can identify $S_{\mathcal{B}}$ with the compact subset $\rho(S_{\mathcal{B}}) = \{s \in S_a: s \equiv 0 \text{ on } \mathcal{M}\}$ of $S_{\mathcal{A}}$. Now, $\eta_{\mathcal{B}}(\tau a) = \{s \in S: s(\tau a) = 1\} = \{s: \rho(s)(a) = 1\}$ so that, in this identification, $\eta_{\mathcal{B}}(\tau a) = S_{\mathcal{B}} \cap \eta_{\mathcal{A}}(a)$; that is, $\rho^{-1}(\eta_{\mathcal{A}}(a)) = \eta_{\mathcal{B}}(\tau a)$. This also means that $\chi_{\mathcal{B}}(\tau a) = \chi_{\mathcal{A}}(a) \circ \rho$, or that $\chi_{\mathcal{B}}(\tau a)$ is but the restriction to $S_{\mathcal{B}}$ of $\chi_{\mathcal{A}}(a)$. We want to study the more general restriction mapping $T: L^{\infty}(\mathcal{A}) \to L^{\infty}(\mathcal{B})$ given by $Tf = f \circ \rho$ and particularly its adjoint T'.

THEOREM 7.1. *T is a norm decreasing linear map of $L^{\infty}(\mathcal{A})$ onto $L^{\infty}(\mathcal{B})$. If $\mathcal{A}$ is σ-complete and $\mathcal{M}$ is a σ-ideal, then T is strictly continuous.*

PROOF. The first claim is obvious. For the second, suppose U is a β-neighborhood of zero in $L^{\infty}(\mathcal{B})$ and that $a_n \nearrow a$ in $\mathcal{A}$. Now $\tau(a_n) \nearrow \tau(a)$ in $\mathcal{B}$, as noted above, so, there is a g, continuous on $S_{\mathcal{B}}$ and vanishing on $E = \eta(\tau(a)) \backslash \bigcup_{n=1}^{\infty} \eta(\tau(a_n))$, such that $\|hg\| \leqslant 1$ puts $h \in U$.

Let $r > 0$ and $B = \{f \in L^{\infty}(\mathcal{A}): \|f\| \leqslant r\}$. Let $K = \{x: |g(x)| \geqslant 1/r\} \subset S_{\mathcal{B}}$. Now $K \cap E = \square$, $\rho(K)$ is compact in $\rho(S_{\mathcal{B}}) \subset S_{\mathcal{A}}$, and, $\rho(K) \cap \eta(a) \backslash \bigcup_{n=1}^{\infty} \eta(a_n) = \square$. Let g_0 be a continuous function on $S_{\mathcal{A}}$, $g_0 = 1$ on $\rho(K)$ and $g_0 = 0$ on the closed set $\eta(a) \backslash \bigcup_{n=1}^{\infty} \eta(a_n)$. Now, if $f \in B_r$ and $\|fg_0\| \leqslant 1/\|g\|$, then $x \in K$ implies

$$|(Tf)(x)g(x)| = |f(\rho(x))g_0(\rho(x))g(x)| \leqslant 1$$

while $x \notin K$ implies

$$|(Tf)(x)g(x)| \leqslant \|f\|(1/r) \leqslant 1.$$

It follows from 3.2 that $T^{-1}(U)$ is a β-neighborhood of zero in $L^{\infty}(\mathcal{A})$.

We turn now to the really nice properties of the adjoint mapping T', which must map $L^1(\mathcal{B})$ onto a subspace of $L^1(\mathcal{A})$ in a 1-1 fashion.

PROPOSITION 7.2. (1) *The range R of T' is the solid linear subspace $\{\hat{\mu} \in L^1(\mathcal{A}): \mathcal{M} \subset |\mu|^{-1}(0)\}$ of $L^1(\mathcal{A})$.*

(2) *For* $\hat{\mu} \in R$ *these are equivalent*: (a) $T'\hat{\nu} = \hat{\mu}$, (b) $\mu = \nu \circ \tau$, (c) $\mu^* = \nu^* \rho^{-1}$ *in the sense of* 4.3 *and* 5.2.

(3) T' *is an isometry.*

PROOF. (1) It is clear that $Tf^+ = (Tf)^+$ for $f \in L^\infty(\mathcal{A})$. If $h \leqslant Tf = f \circ \rho$, let g be any continuous extension of $h \circ \rho^{-1}$ (defined on $\rho(S_{\mathfrak{B}})$) to $S_{\mathcal{A}}$ such that $g \leqslant f$. Then $Tg = h$. From this, $T'\hat{\nu}^+ = (T'\hat{\nu})^+$ for all $\hat{\nu} \in L^1(\mathfrak{B})$. Hence, $T'|\hat{\nu}| = |T'\hat{\nu}|$ as well.

If $\hat{\mu} = T'\hat{\nu} \geqslant 0$ and $a \in \mathfrak{M}$, then $\mu(a) = \hat{\mu}(\chi_{\mathcal{A}}(a)) = \langle T'\hat{\nu}, \chi_{\mathcal{A}}(a)\rangle = \langle \hat{\nu}, \chi_{\mathcal{A}}(a) \circ \rho\rangle = \langle \hat{\nu}, \chi_{\mathfrak{B}}(\tau a)\rangle = \nu(\tau a) = \nu(0) = 0$. Hence $\mathfrak{M} \subset |\mu|^{-1}(0)$ for any $\hat{\mu} = T'\hat{\nu}$. Conversely, if $\mathfrak{M} \subset |\mu|^{-1}(0)$, then $\nu^\pm(\tau a) = \mu^\pm(a)$ are well-defined finitely additive measures on $\mathfrak{B}$. Because $\tau(\bigvee a_n) = \bigvee \tau(a_n)$ both define functionals $\hat{\nu}^\pm \in L^1(\mathfrak{B})$ by 4.3, and with $\hat{\nu} = \hat{\nu}^+ - \hat{\nu}^-$, we have $(T'\hat{\nu})\chi(a) = \hat{\mu}(\chi(a))$ so that $\hat{\mu} = T'\hat{\nu}$. From this it also follows that R is solid and (1) is proven.

(2) In the paragraph above we have already shown that (a) implies (b). Now (b) implies that $\hat{\mu} \circ \chi_{\mathcal{A}} = \hat{\nu} \circ \chi_{\mathfrak{B}} \circ \tau$, or $\langle \hat{\mu}, \chi_{\mathcal{A}}(a)\rangle = \langle \hat{\nu}, T\chi_{\mathcal{A}}(a)\rangle$. Hence $T'\hat{\nu}$ agrees with $\hat{\mu}$ on $\mathbb{S}(\mathcal{A})$ and hence on $L^\infty(\mathcal{A})$ and (a) holds. The equivalence with (c) is just the change of variable formula in integration.

(3) $\|T'\hat{\nu}\|_1 = |\hat{\mu}|(1) = |\mu^*|(S_{\mathcal{A}}) = |\nu^*|(S_{\mathfrak{B}}) = |\hat{\nu}|(1) = \|\hat{\nu}\|_1$.

Although (3) is a little surprising it is intuitively evident from the observation that $\hat{\mu} \in R$ iff the support of μ^* lies in $\rho(S_{\mathfrak{B}})$. We record this in

COROLLARY 7.3. *Let* $S_\mu = \{s \in S_{\mathcal{A}}: |\mu^*|(\eta(a)) > 0$ *whenever* $s(a) = 1\}$ *be the support of* μ^* *in* $S_{\mathcal{A}}$. *Then* $\hat{\mu} \in R$ *iff* $S_\mu \subset S_{\mathfrak{B}}$.

PROOF. We will use the notation $\rho(S_{\mathfrak{B}})$ in the proof and want to show that $\hat{\mu} \in R$ iff $S_\mu \subset \rho(S_{\mathfrak{B}})$. Suppose that $\hat{\mu} = T'\hat{\nu}$, and that $s' \in S_\mu$. If $s \notin \rho(S_{\mathfrak{B}})$ there is an $a \in \mathcal{A}$ with $s' \in \eta(a)$ and $\eta(a) \cap \rho(S_{\mathfrak{B}}) = \square$. Hence $|\mu^*|(\eta(a)) = |\nu^*|\rho^{-1}(\eta(a)) = 0$, a contradiction.

If conversely, $S_\mu \subset \rho(S_{\mathfrak{B}})$, then $a \in \mathfrak{M}$ implies that $s(a) = 0$ for all $s \in S_\mu$. If $|\mu|(a) > 0$, then $|\mu^*|(\eta(a)) > 0$ so that for any $s_0 \in \eta(a)$ we would have $s_0 \in S_\mu$ and then $s_0(a) = 0$, a contradiction. Hence $a \in M$ implies $|\mu|(a) = 0$ and $\hat{\mu} \in R$ by 7.2(a), completing the proof.

Of course, $S_\mu \subset S_{\mathfrak{B}}$ is the topological dual of the algebraic statement $\mathfrak{M} \subset |\mu|^{-1}(0)$. We also note that $T'L^1(\mathfrak{B}) = R$ is also a band in $L^1(\mathcal{A})$ (7.2(a)).

Our reason for the above consideration is the study of a set $H \subset L^1(\mathcal{A})$ of functionals on $\mathcal{A}$ by way of the σ-ideal $\mathfrak{M} = \bigcap_{\hat{\mu} \in H} |\mu|^{-1}(0)$; this is the obvious analogue of our earlier remarks. For such an $\mathfrak{M}$ we can say a little more.

PROPOSITION 7.4. $S_{\mathfrak{B}} = \overline{\bigcup_{\hat{\mu} \in H} S_\mu}$ *and* R *contains the* $\|\ \|_1$*-closed linear span of* H *in* $L^1(\mathcal{A})$.

PROOF. Already $\overline{\bigcup S_\mu} \subset S_{\mathfrak{B}}$ from 7.3. If $s \in S_{\mathfrak{B}} \backslash \overline{\bigcup S_\mu}$, then there is an

$a \in \mathcal{A}$ for which $\eta(a) \cap \overline{\cup S_\mu} = \square$, whence $|\mu^*|(\eta(a)) = |\mu|(a) = 0$ for all $\mu \in H$. But then, $a \in \mathfrak{M}$, $s \in S_{\mathfrak{B}}$ and $s(a) = 1$, a contradiction. The rest is clear.

It follows from the general theory of linear maps on locally convex spaces [27] that T' is not only $\| \|_1$-continuous but also weakly $(\sigma(L^1, L^1{}'))$- and weak* $(\sigma(L^1, L^\infty))$-continuous. Of course, being an isometry T'^{-1} is also $\sigma(R, R')$-$\sigma(L^1(\mathfrak{B}), L^1(\mathfrak{B})')$-continuous and hence, T'^{-1} is also weakly continuous. More importantly,

PROPOSITION 7.5. *For $\hat{\mu} \in R$ and $g \in L^\infty(\mathfrak{B})$, $\langle T'^{-1}\hat{\mu}, g\rangle = \langle \hat{\mu}, f\rangle$ where $Tf = g$, and T' is bicontinuous for weak, weak* and $\| \|_1$ topologies on $L^1(\mathcal{A})$ and R.*

PROOF. Since T is onto, there is an $f \in L^\infty(\mathcal{A})$ for which $Tf = g$. That is, $f \circ \rho = g$ on $S_{\mathfrak{B}}$. If $\hat{\nu} = T'^{-1}\hat{\mu}$, then (7.2) $\mu^* = \nu^* \rho^{-1}$ and hence $\langle T'^{-1}\hat{\mu}, g\rangle = \int_{S_{\mathfrak{B}}} g\, d\nu^* = \int_{S_{\mathfrak{B}}} f \circ \rho\, d\nu^* = \int_{S_{\mathcal{A}}} f\, d\mu^* = \langle \hat{\mu}, f\rangle$. The rest now follows.

We are ready to prove the principal theorem of this section.

THEOREM 7.6. *Let $H \subset L^1(\mathcal{A})$ be weak* countably compact and let $A \subset \mathcal{A}$ be a pairwise disjoint subset of $\mathcal{A}$. Then*

(a) *$\bigcup_{\hat{\mu} \in H}\{a \in A: |\mu|(a) \geqslant \delta > 0\}$ is finite.*

(b) *Let J denote the weak* closed convex hull of H and let*

$$\mathfrak{M} = \bigcap_{\hat{\mu} \in J} |\mu|^{-1}(0).$$

Then $\mathcal{A}/\mathfrak{M}$ is complete and meets the countable chain condition and the strict topology β on $L^\infty(\mathcal{A}/\mathfrak{M})$ is Hausdorff.

PROOF. (a) Suppose this union contains denumerably many distinct elements $\{a_n\}$. For each n, let f_n be the lattice supremum of $\{\chi(a_k): k \geqslant n\}$. Clearly, $f_n = \chi_{\overline{\cup_{k=n}^\infty \eta(a_k)}}$ and for any $\hat{\mu} \in L^1(\mathcal{A})$, $\hat{\mu}(f_n) = \sum_{k=n}^\infty \mu(a_k)$ by 4.2. Hence, for $\hat{\mu} \geqslant 0$, $\hat{\mu}(f_n) \searrow 0$.

By 4.9, H is β-equicontinuous. Since β is locally solid, $\{|\mu|: \mu \in H\}$ is also β-equicontinuous hence weak* relatively compact. Hence $|\hat{\mu}|(f_n) \searrow 0$ uniformly over $\hat{\mu} \in H$ by Dini's Theorem. But this contradicts the assumption that $|\mu|(a_n) \geqslant \delta$ for some $\hat{\mu} \in H$ and each n.

(b) To complete the argument we note that since H is equicontinuous, $H \subset W^0$, where W is a β-neighborhood of zero. Hence $J \subset W^0$ and J is w^*-compact.

Now $K = T'^{-1}(J)$ is weak* compact in $L^1(\mathfrak{B})$ where $\mathfrak{B} = \mathcal{A}/\mathfrak{M}$ by 7.5. From the first part of our argument, if $B \subset \mathfrak{B}$ is pairwise disjoint, then $\bigcup_{\hat{\nu} \in K}\{b: |\nu|(b) > 0\}$ is at most countable. But if $|\nu|(b) = 0$ for all $\hat{\nu} \in K$ it follows from the definition of $\mathfrak{M}$ that b is the zero element of $\mathfrak{B}$. Hence $\mathfrak{B}$ has c.c.c. and for the same reason β is Hausdorff. Since a β-complete Boolean algebra with countable chain condition is complete we are done.

REMARK. It is interesting (and important) that the first part of the above proof avoids the issue of weak countable distributivity in $\mathcal{A}$ (see [18]) by virtue of the defining countable additivity property of a $\hat{\mu} \in L^1(\mathcal{A})$.

COROLLARY 7.7. *For H a bounded subset of $L^1(\mathcal{A})$, these are equivalent:*
(1) *H is $\sigma(L^1(\mathcal{A}), \mathbb{S}(\mathcal{A}))$-countably compact.*
(2) *H is $\sigma(L^1(\mathcal{A}), L^\infty(\mathcal{A}))$-countably compact.*
(3) *H is $\sigma(L^1(\mathcal{A}), L^1(\mathcal{A})')$-relatively compact.*
(4) *H is β-equicontinuous.*

PROOF. Now (3) $\to$ (2) $\to$ (1) and (1) $\to$ (2) since H is bounded and $\mathbb{S}(\mathcal{A})$ is norm dense in $L^\infty(\mathcal{A})$. From 4.9, (2) $\to$ (4). It remains to obtain (4) $\to$ (3). By the theorem above, with J and $\mathfrak{M}$ so defined, $\mathfrak{B}$ meets the countable chain condition, and hence by 6.3 $L^1(\mathfrak{B})' = L^\infty(\mathfrak{B})$. So the weak and weak* topologies coincide on $T'^{-1}(H)$. But this set is weak* relatively compact, T' is weakly continuous, and (3) follows.

Clearly we could more generally argue that if $H \subset L^1(\mathcal{A})$ and the associated $\mathfrak{B}$ has c.c.c., then the weak and weak* topologies agree on H.

COROLLARY 7.8. *A sequence $\{\hat{\mu}_n\} \subset L^1(\mathcal{A})$ is weakly convergent iff it is weak* convergent.*

A most important corollary to 7.6 now follows from 6.9. It, along with 6.9, lead us directly into §8 and the important topic of Boolean measure algebras where, using 7.9, we will refine 4.11 a bit further.

COROLLARY 7.9. *The algebra $\mathcal{A}/\mathfrak{M}$ of 7.6 admits a strictly positive normalized measure.*

8. **Boolean measure algebras.** A pair $(\mathcal{A}, \mu)$ where μ is a strictly positive normalized measure μ on $\mathcal{A}$ is called a Boolean measure algebra [16]. These are a major reason for a role for Boolean algebras in measure theory, for if $\bar{\mu}$ is a nonnegative finite measure on a measure space (X, Σ), then $\mathcal{A} = \Sigma/\bar{\mu}^{-1}(0)$ along with $\mu[E] = \bar{\mu}(E)/\bar{\mu}(X)$ is a Boolean measure algebra. More generally, if $\hat{\mu} \in L^1(\mathcal{A})$ and $\mathfrak{M} = |\bar{\mu}|^{-1}(0)$ then $\mathfrak{B} = \mathcal{A}/\mathfrak{M}$ along with the measure ν on $\mathfrak{B}$ such that $T'\nu = \mu$ (7.2(b)) is a Boolean measure algebra. By Corollary 7.9, so can $\mathcal{A}/\mathfrak{M}$ (where $\mathfrak{M}$ is as in 7.6(b)) be made into the same, and by 6.8, $L^1(\mathcal{A})' = L^\infty(\mathcal{A})$ for σ-complete c.c.c. $\mathcal{A}$ iff $\mathcal{A}$ can be made into such an algebra. The $L^1(\mathcal{A})$-space for such an algebra is a true analogue of ordinary L^1-spaces and the topology β turns out to be somewhat familiar on the associated $L^\infty(\mathcal{A})$.

We wish to quickly remark that the results of this section can be had for the more general σ-finite analogue–there is a sequence $a_n \nearrow e$ such that $\mu(a_n) < \infty$, rather than $\mu(e) = 1$–but no additional insight is gained and we consider only the case $\mu(e) = 1$.

Throughout this section we suppose $\mathcal{Q}$ is σ-complete. If $(\mathcal{Q}, \mu)$ is given it then follows, as has already been noted, that $\mathcal{Q}$ has c.c.c. and is complete and β is Hausdorff, hence $L^1(\mathcal{Q})' = L^\infty(\mathcal{Q})$. At the end of this section we will return again to this last equality. Our intent in the section is to draw out the special properties of our principal objects of study–the topology β and its dual $L^1(\mathcal{Q})$–which flow from the assumed existence of a μ on $\mathcal{Q}$ with $\mu(a) > 0$ for all $a \neq 0$, and then see what these results say about the more general setting of previous sections. We begin by seeing how β-equicontinuity is uniform absolute continuity with respect to μ.

THEOREM 8.1. *A subset $H \subset L^1(\mathcal{Q})$ is weak* countably compact (hence β-equicontinuous) iff H is $\| \|_1$-bounded and $\varepsilon > 0$ implies there exists $\delta > 0$ such that $\mu(a) < \delta$ implies $|\nu|(a) < \varepsilon$ for all $\hat{\nu} \in H$.*

PROOF. Suppose this were not true and inductively define a decreasing sequence $b_n \in \mathcal{Q}$ such that $\mu(b_n) < 1/n$ and $|\nu_n|(b_n) \geq \varepsilon$ for some $\hat{\nu}_n \in H$.

Let $a_n = e - b_n$. Then $\mu(a_n) = \mu(e) - \mu(b_n) \to \mu(e) = 1$. If $a = \sup a_n$, then $\mu(a) = 1$ since μ is countably additive. Since μ is strictly positive $e = a = \sup a_n$.

But now, by 4.11, $\{|\hat{\nu}| \circ \chi \colon \hat{\nu} \in H\}$ is uniformly countably additive. Hence $|\nu|(b_n) = |\nu|(e) - |\nu|(a_n) < \varepsilon$ for all $\hat{\nu} \in H$, if n is sufficiently large, a contradiction.

To finish the necessity, recall that $L^1(\mathcal{Q})$ is a subset of the Banach space dual of $L^\infty(\mathcal{Q})$ (3.2(a)), hence H is $\| \|_1$-bounded.

Conversely, if the latter conditions of the theorem hold, and if $a_n \nearrow a$ in $\mathcal{Q}$, then given $\varepsilon > 0$, choose δ according to the hypothesis, and then N such that $n > N$ makes $\mu(a - a_n) < \delta$. Then $|\nu|(a - a_n) < \varepsilon$ for all $\hat{\nu} \in H$; that is, H is uniformly countably additive. Since H is bounded it now follows from 4.11 that H is β-equicontinuous hence weak* relatively compact, completing the proof.

Theorem 8.1 is of course the Dunford-Pettis criteria for weak compactness in L^1 of a finite measure space [10, 4.21.2]. For the σ-finite case one need only add the condition: $\varepsilon > 0$ implies there exist $b \in \mathcal{Q}$ such that $\mu(b) < \infty$ and $|\nu|(e - b) < \varepsilon$ for all $\hat{\nu} \in H$, where μ is only supposed to be σ-finite (and strictly positive).

The method of proof for the converse above also allows a characterization of functionals on $L^\infty(\mathcal{Q})$ which lie in $L^1(\mathcal{Q})$ special to Boolean measure algebras.

PROPOSITION 8.2. *A linear functional $\hat{\mu}$ on $L^\infty(\mathcal{Q})$ lies in $L^1(\mathcal{Q})$ iff $\hat{\nu}$ is bounded and $\varepsilon > 0$ implies there is a $\delta > 0$ such that $\mu(a) < \delta$ implies $|\hat{\nu}|(\chi(a)) < \varepsilon$.*

For as in the proof above, this condition forces $|\hat{\nu}| \circ \psi$ to be a countably additive measure on $\mathcal{Q}$, whence $\hat{\nu} \in L^1(\mathcal{Q})$ by 4.4.

As another corollary of 8.1 we have an extension of the result in [9, 9.2].

THEOREM 8.3. *Let $\mathcal{C}$ be any σ-complete Boolean algebra. A set $H \subset L^1(\mathcal{C})$ is weak* countably compact in $L^1(\mathcal{C})$ iff H is bounded and there is a $\hat{\mu} \in L^1_+(\mathcal{C})$ such that H is uniformly absolutely continuous with respect to μ (in the sense of 8.1).*

PROOF. The sufficiency of this latter condition is clear from 4.11 and the proof of 8.1. For necessity, let $\mathfrak{B}$ be as in 7.6, ν strictly positive on $\mathfrak{B}$ by 7.9, and $\hat{\mu} = T'\hat{\nu}$, by 7.2. For a given $\varepsilon > 0$, choose $\delta > 0$ such that $\nu(b) < \delta$ implies $|\gamma|(b) < \varepsilon$ for all $\gamma \in T'^{-1}(H)$, according to 8.1 applied to $L^1(\mathfrak{B})$. If $\mu(a) < \delta$ and $b = \tau(a)$, then by 7.2, for any $\hat{\gamma}' \in H$ with $T'\hat{\gamma} = \hat{\gamma}'$ we have $\hat{\gamma} \in T'^{-1}(H)$, and $|\gamma'|(a) = |\gamma|(\tau a) < \varepsilon$ since $\nu(\tau a) < \delta$, completing the proof.

From 8.3 we can obtain an even more surprising corollary. In [17], Henry and Taylor showed that their topology $\bar{\beta}$ was a strong Mackey topology under conditions equivalent to c.c.c. The same result holds here, for if $\mathcal{C}$ is σ-complete with c.c.c. then $L^1(\mathcal{C}) = N(S) = L(\mathcal{C})$ (6.3 and 6.4) and then $\bar{\beta} = \beta$ (5.4) is a strong Mackey space (4.9). With the aid of the results just above we are able to show that $\bar{\beta}$ as defined herein is always Mackey. This is a sharp departure from the analogue in (C_b, M) duality where the question of when β is the Mackey topology for the dual pair (C_b, M_τ) is not yet settled (e.g., see [28]).

THEOREM 8.4. *Let $\mathcal{C}$ be a complete Boolean algebra. A weak* countably compact subset $H \subset L(\mathcal{C})$ is $\bar{\beta}$-equicontinuous.*

PROOF. Let $\hat{\mu}$ be the functional in $L^1(\mathcal{C})$ guaranteed by 8.3 where $T'\hat{\nu} = \hat{\mu}$ and $\hat{\nu} \in L^1(\mathfrak{B})$. It suffices to prove that if $a_\alpha \nearrow a$ in $\mathcal{C}$, then there is an α_0 such that $\alpha \geqslant \alpha_0$ implies $|\hat{\gamma}| \circ \chi(a - a_\alpha) = |\gamma|(a - a_\alpha) < \varepsilon$ for all $\hat{\gamma} \in H$. From 8.3 it suffices to prove that $\mu(a - a_\alpha) \to 0$ in α. Now $\{\tau(a_\alpha)\}$ is an increasing net in $\mathfrak{B}$ (7.6) with supremum $b = \tau(c)$ for some $c \in \mathcal{C}$ and such that b is the supremum of some increasing countable subset of $\{\tau(a_\alpha)\}$, say $\{b_n\}$ (for $\mathfrak{B}$ has c.c.c.). Since $b_n \nearrow b$, there is an n_0 such that $\nu(b - b_{n_0}) < \varepsilon$. Let $b_{n_0} = \tau(a_{\alpha_0})$. Then by 7.2(b) $\mu(c - a_{\alpha_0}) = \nu(b - b_{n_0}) < \varepsilon$. We claim $\mu(c) = \mu(a)$. For, $\eta(c) \cap S_{\mathfrak{B}} = \eta(a) \cap S_{\mathfrak{B}}$, (the notation precedes 7.1) and by 7.3 the support S_μ of μ is a subset of $S_{\mathfrak{B}}$. Hence $\mu(c) = \mu^*(\eta(c)) = \mu^*(\eta(c) \cap S_{\mathfrak{B}}) = \mu^*(\eta(a)) = \mu(a)$. Hence for $\alpha \geqslant \alpha_0$, $\mu(a - a_\alpha) \leqslant \mu(a - a_{\alpha_0}) = \mu(c - a_{\alpha_0}) < \varepsilon$ and we are done.

We turn now to a description of β-convergence in $L^\infty(\mathcal{C})$ and some comparisons with the β-theory of dual pairings (C_b, M). If X is a locally compact Hausdorff space, β is the finest locally convex topology on $C_b(X)$ agreeing with the compact-open topology on all bounded subsets B_r of $C_b(X)$ [26, 2.3]. On the other hand if X is completely regular, we know of no criteria for strict convergence in terms of the underlying space X, save for the topology

β_0 [26]. Of course β herein is the analogue of β_1 [26] for pairings (C_b, M). In general, we can say no more for β-convergence on $L^\infty(\mathcal{A})$ than for β_1 in $C_b(X)$ (other than results like 5.3) except under the conditions of this section, whereupon matters are analogous to the locally compact case.

Let $(\mathcal{A}, \mu)$ be a Boolean measure algebra, let $\hat{\mu}(f) = \int_S f\,d\mu^*$, and define $d(f,g) = \inf\{\alpha\colon \mu^*(\{x\colon |f(x) - g(x)| \geqslant \alpha\}) \leqslant \alpha\}$ for $f, g \in L^\infty(\mathcal{A})$. (For $f \in L^\infty(a)$, let $\{f \geqslant \alpha\} = \{x \in S\colon f(x) \geqslant \alpha\}$.) For $f \neq g$, $d(f,g) > 0$ since μ is strictly positive, and d is a metric on $L^\infty(\mathcal{A})$. Naturally one would call the d-topology on $L^\infty(\mathcal{A})$ the topology of convergence in measure. Note that an alternative definition of d is $d(f,g) = \inf\{\alpha\colon \mu(\eta(a)) \leqslant \alpha$ where $a = e - b$ and $\eta(b) = \overline{\{|f-g| < \alpha\}}\}$.

In [6], Dazord and Jourlin study $L^1(X, \mathfrak{B}_0(X), \mu)$ as the dual of $L^\infty(X, \mathfrak{B}_0(X), \mu)$ under the topology of convergence in measure localized on all sets B_r where X is completely regular, and μ is a tight Borel measure on the Borel sets $\mathfrak{B}_0(X)$ of X, and remarked on the analogy with β_0 for the $(C_b(X), M_t(X))$ duality. We obtain this result for abstract measure spaces (X, Σ, μ) and show that the topology is indeed a strict topology, as much like β or β_1 [26], as β_0.

THEOREM 8.5. *β is the finest locally convex topology on $L^\infty(\mathcal{A})$ agreeing with the topology of d-convergence on all sets $B_r = \{f\colon \|f\| \leqslant r\} \subset L^\infty(\mathcal{A})$, $r > 0$.*

PROOF. We show first that d-convergence implies β-convergence on the sets B_r. It is enough to show that if $H \subset L^1$ is β-equicontinuous and $\{f_n\} \subset B_r$ such that $d(f_n, f) \to 0$ and $f \in B_r$, then $|\hat{\nu}(f_n - f)| \to 0$ uniformly over $\hat{\nu} \in H$. We can assume $\|\hat{\nu}\|_1 \leqslant 1$ for all $\hat{\nu} \in H$.

Suppose that E is a Borel set in X for which $\mu^*(E) < \delta$. Then since μ^* is regular there is an open set $U \supset E$ such that $\mu(U) < \delta$. But then there is a subset $B \subset \mathcal{A}$ such that $U = \bigcup_B \eta(b)$. Let $a = \sup B$. Since $\mathcal{A}$ has c.c.c., $a = \sup a_n$ where $\{a_n\}$ is increasing and $\eta(a_n) \subset \bigcup \eta(b) = U$. Hence, $\eta(a) \supset U \supset \eta(a_n)$ for all n, and by 4.3 $\mu(a) = \sup \mu(a_n)$, and from this, $\mu(a) = \mu^*(U) < \delta$.

Now since H is β-equicontinuous, 8.1 implies that if $\varepsilon > 0$, there is a $\delta < \varepsilon/2$, $\delta > 0$, such that $\mu(a) < \delta$ implies $|\nu|(a) < \varepsilon/4r$ for all $\hat{\nu} \in H$. If n_0 is chosen so that $n \geqslant n_0$ implies $d(f_n, f) < \delta$, then there is a $\delta_1 < \delta$ such that $\mu^*(|f_n - f| \geqslant \delta_1) \leqslant \delta_1$. With $E = \{|f_n - f| \geqslant \delta_1\}$, for a fixed n, choose $a \in \mathcal{A}$ as above so that $\mu(a) < \delta_1$ and $E \subset \eta(a)$. Then $|\nu^*|(E) \leqslant |\nu|(a) < \varepsilon/4r$ for all $\hat{\nu} \in H$. Hence, for this $n > n_0$,

$$|\hat{\nu}(f_n - f)| \leqslant \int_E |f_n - f|\,d|\nu^*| + \int_{S/E} |\delta_n - f|\,d|\nu^*|$$

$$\leqslant 2r(\varepsilon/4r) + \delta_1 \|\hat{\nu}\|_1 < \varepsilon$$

for all $\hat{\nu} \in H$, and this is independent of n. Hence $f_n \to^{\beta} f$.

Conversely, we claim that β-convergence implies d-convergence. For any $f \in L^{\infty}(\mathcal{C})$, $|\mu^*|(|f| \geqslant \varepsilon) \leqslant 1/\varepsilon\hat{\mu}(|f|)$. Hence if $f_\alpha \to^{\beta} 0$, then since $|f_\alpha| \to^{\beta} 0$ by 5.5, then choosing α_0 so that $\alpha \geqslant \alpha_0$ implies $|\hat{\mu}|(|f_\alpha|) < \varepsilon^2$ we obtain $|\mu^*|(|f_\alpha| \geqslant \varepsilon) < \varepsilon$ whence $d(f_\alpha, 0) < \varepsilon$ completing the proof.

REMARK. The previous paragraph shows that we have a usable locally convex substitute for the nonlocally convex topology of convergence in measure, and this result now extends to ordinary measure spaces.

COROLLARY 8.6. *The topology β_∞ of* 4.8 *is the finest locally convex topology on $L^{\infty}(X, \Sigma, \bar{\mu})$ agreeing with convergence in measure on $\| \|_\infty$-bounded sets. The β_∞ continuity of linear maps on $L^{\infty}(X, \Sigma, \bar{\mu})$ is equivalent to continuity under convergence in measure on bounded sets.*

PROOF. Recall the map θ: $L^{\infty}(X, \Sigma, \mu) \to L^{\infty}(\mathcal{C})$ of 4.8. We will show that it is a metric isometry for the metrics of convergence in measure on the respective spaces. A word of caution: in this case we are dealing with a σ-finite measure space and σ-finite Boolean measure algebra. Recalling the remark following 8.1, it should be clear what the necessary modifications are and we will make the proof only for $\bar{\mu}(X) < \infty$.

According to our note preceding 4.5, if $s \in S_{\mathcal{C}}$ where $\mathcal{C} = \Sigma/\bar{\mu}^{-1}(0)$, then $[\theta f](s) = \sup\{\alpha: s([f \geqslant \alpha]) = 1\}$. It follows from this that for $f > 0$ everywhere on X, $\{\theta f \geqslant \alpha\} \supset \eta[f \geqslant \alpha] \supset \eta[f > \alpha] \supset \{\theta f > \alpha\}$. Hence,

$$\mu^*(\{\theta f \geqslant \alpha\}) \geqslant \bar{\mu}(\{f \geqslant \alpha\}).$$

Suppose that $\mu^*(\{\theta f \geqslant \alpha\}) > \delta + \bar{\mu}(\{f \geqslant \alpha\})$. Since

$$\{f \geqslant \alpha\} = \bigcap_n \{f > \alpha - 1/n\}$$

there is an n_0 such that $n \geqslant n_0$ implies

$$\bar{\mu}(\{f > \alpha - 1/n\}) < \mu^*(\{\theta f \geqslant \alpha\}) - \delta.$$

But $\bar{\mu}(\{f > \alpha - 1/n_0\}) \geqslant \mu^*(\{\theta f > \alpha - 1/n\})$ from our earlier remark. Hence

$$\mu^*(\{\theta f > \alpha - 1/n\}) < \mu^*(\{\theta f \geqslant \alpha\}) - \delta$$

for all $n \geqslant n_0$ a contradiction. Thus $\bar{\mu}(\{f \geqslant \alpha\}) = \mu^*(\{\theta f \geqslant \alpha\})$. It now follows that $d_\infty(f, g) = d(\theta f, \theta g)$ where d_∞ is a metric for convergence in measure on $L^{\infty}(X, \Sigma, \bar{\mu})$.

The assertions of the corollary now quickly follow from this, the definition of β_∞, and 8.5.

Corollary 8.4 is another justification for introducing the topology β, for the fact that β, i.e. β_∞, is natural and familiar on ordinary measure spaces $L^{\infty}(X, \Sigma, \bar{\mu})$, we believe, enhances the more general β on $L^{\infty}(\mathcal{C})$.

We conclude with some results related to the Radon-Nikodým theorem and complete the circle of ideas beginning in §6 with a sketch of a direct proof of 6.9(b) and (c).

For $f \in L^\infty$ and $\hat{\nu} \in L^1$ let $(f \cdot \hat{\nu})(g) = \hat{\nu}(fg)$. It follows from 4.12 that $f \cdot \hat{\nu} \in L^1$. Let P: $L^\infty(\mathcal{A}) \to L^1(\mathcal{A})$ be defined by $P_{\hat{\nu}} f = f \cdot \hat{\nu}$. As an extension of 4.10:

PROPOSITION 8.7. *$P_{\hat{\nu}}$ is β-$\| \|_1$-continuous.*

PROOF. If $\|g\| \leqslant 1$ in $L^\infty(\mathcal{A})$, and $f_\alpha \to^\beta 0$, then

$$|P(f_\alpha)(g)| = |\hat{\nu}(f_\alpha g)| \leqslant \|g\| \, |\hat{\nu}|(|f_\alpha|)$$

which converges to zero since $|\hat{\nu}| \in L^1$ by 3.3 and 5.5.

Suppose now that $(\mathcal{A}, \mu)$ is a measure algebra and $P = P_{\hat{\mu}}$. Since μ is strictly positive, $\|Pf\|_1 = 0$ implies $\hat{\mu}(f^2) = 0$ and hence $f = 0$ and P is 1-1. Let $M = P(L^\infty)$.

THEOREM 8.8. (a) *P is a β-continuous* 1-1 *mapping of $L^\infty(\mathcal{A})$ into $L^1(\mathcal{A})$ such that if $f \wedge g = 0$ in $L^\infty(\mathcal{A})$, then $\|P(f+g)\|_1 = \|Pf\|_1 + \|Pg\|_1$.*

(b) *M is $\| \|_1$-dense in $L^1(\mathcal{A})$.*

(c) *On each set $P(B_r) \subset L^1(\mathcal{A})$, P^{-1} is $\| \|_1$-β-continuous.*

PROOF. (a) The first claim is 8.7. For the second we need only note that the Borel measures representing Pf and Pg as functionals on $C(S) = L^\infty(\mathcal{A})$ have disjoint support and that the norm of L^1 is the variation norm on these representing measures.

(b) Since μ is strictly positive, the orthogonal complement (or polar) of M in L^∞ must be $\{0\}$. But by 6.3 and our earlier remarks $L^{1\prime} = L^\infty$. Hence M is weakly dense in L^1 and so $\| \|_1$-dense.

(c) For suppose that $\|f_n\| \leqslant r$ and $\|f\| \leqslant r$ and $\|Pf_n - Pf\|_1 \to 0$. Now

$$\|P(f_n - f)\|_1 = \int_S |f_n - f| \, d\mu^* \geqslant \varepsilon\mu^*(\{|f_n - f| \geqslant \varepsilon\})$$

so that $d(f_n, f) \to 0$. By 2.3, $f_n \to f$ in the strict topology β.

REMARK. Part (b) tells us of course that the Radon-Nikodým correspondence is a β-$\| \|_1$ homeomorphism at least on certain subsets. Also, we see that $\|f\|' = \|f \cdot \hat{\mu}\|_1$ is a norm on $L^\infty(\mathcal{A})$ defining β on each bounded set B_r.

The problem of characterizing Boolean algebras which admit a strictly positive finitely additive (or more generally countably additive), measure is an old one. The first results are due to Maharam [19], then Kelley [18] and more recently Gaifman [13]. These characterizations are (properly) directly in terms of the algebra $\mathcal{A}$. We have some mild results in terms of $L^1(\mathcal{A})$ which delve into the earlier characterization (6.9) somewhat further. The easiest is

THEOREM 8.9. *If $\mathfrak{A}$ is a Boolean algebra for which there is a mapping* $P: L^\infty(\mathfrak{A}) \to L^1(\mathfrak{A})$ *satisfying* 8.8(a), *then $\mathfrak{A}$ is a Boolean measure algebra with strictly positive countably additive measure* $\mu(a) = \|P\chi(a)\|_1$.

To get a little deeper we have a multiplier result which answers the question: "How can one tell if a $\hat{\nu} \in L^1(\mathfrak{A}, \mu)$ comes from an $f \in L^\infty$?"

PROPOSITION 8.10. $\hat{\nu} \in M = PL^\infty$ *iff the operator* T: $PL^\infty \to L^1$ *by* $T(Pf) = f \cdot \hat{\nu}$ *has a bounded extension to all of* L^1, *and*, $\hat{\nu} = (T'1) \cdot \hat{\mu}$.

PROOF. Since μ is strictly positive, T is well defined. If $\hat{\nu} \in M$, with $\hat{\nu} = Pg$, then $\|Pf\|_1 \leqslant 1$ implies $\|T(Pf)\|_1 = \|P(fg)\|_1 \leqslant \|g\|$. Hence T is a bounded linear operator on the dense subspace M into the B-space L^1 and so has a unique bounded extension to all of L^1.

Conversely, if T has a bounded extension, then using 6.9, T': $L^\infty \to L^\infty$ so that $g = T'1 \in L^\infty$. Now, for any $f \in L^\infty$, $\langle f, \hat{\nu}\rangle = \langle 1, f \cdot \hat{\nu}\rangle = \langle 1, T(Pf)\rangle = \langle T'1, Pf\rangle = \langle g, Pf\rangle = \langle f, Pg\rangle$, hence $\hat{\nu} = Pg \in M$.

This last result makes it clear why an absolutely continuous measure "comes from an L^1-function" but a bounded linear functional on L^1 "comes from an L^∞-function". For suppose from the moment that ϕ is a bounded functional on L^1 and set $\hat{\nu}(f) = \phi(f \cdot \hat{\mu})$ where $(\mathfrak{A}, \mu)$ is a measure algebra. Then, $\hat{\nu} \in M$, for $\|T(Pf)\| \leqslant \|\phi\| \, \|Pf\|$ where $T(Pf) = \hat{\nu}(f)$. By 8.10 $\hat{\nu} = g \cdot \hat{\mu}$ and on the dense subset M, $\phi = \hat{\mu}_g$ so that on all of $L^1(\mathfrak{A})$, $\phi(\hat{\gamma}) = \hat{\gamma}(g)$; in fact, $\phi = T'1$.

These observations bring us full circle from the beginnings of §6; note that §6 depends on Kakutani's L-space representation theorem and the Radon-Nikodým theorem for ordinary L^1-spaces [7]. Let us give a quick indication of a direct argument for 6.3 (whose hypotheses are equivalent to assuming that $\mathfrak{A}$ is a measure algebra (6.9)), which at the same time represents elements of $L^1(\mathfrak{A})$ as functions on S; this of course is a departure from our attitude that S is a transient tool and a generalization of a standard result about ordinary L^1-spaces, but seems basic and important enough for inclusion herein. Essentially we will see how the standard proof of the Radon-Nikodým theorem carries over to this setting and at the same time yields a proof of 6.3. We need two familiar lemmas which are easy to obtain for σ-complete algebras with c.c.c.

LEMMA 8.11. *If* $\hat{\mu}, \hat{\nu} \neq 0$ *in* L^1_+, *then there is an* $\alpha > 0$ *such that* $(\hat{\nu} - \alpha\hat{\mu})^+ \neq 0$.

LEMMA 8.12. *If* $\hat{\nu} \in L^1$ *and* $\hat{\nu}^+ \neq 0$, *then there is an* $a \in \mathfrak{A}$ *with* $\hat{\nu}_a \geqslant 0$.

Suppose that $\{g_n\} \subset \mathbb{S}(\mathfrak{A})^+$ is an increasing sequence with the lower-semicontinuous function $g(x) = \sup g_n(x)$, $x \in S$, its pointwise supremum. The functional $\hat{\nu}(f) = \mu^*(gf)$, for $\hat{\mu} \in L^1_+(\mathfrak{A})$ is defined on $L^\infty(\mathfrak{A})$. If, additionally, there is a $\hat{\gamma} \in L^1$ such that $g_n \cdot \hat{\mu} \leqslant \hat{\gamma}$ for all n, then $\hat{\nu} \leqslant \hat{\gamma}$ and it follows from 4.3 that $\hat{\nu} \in L^1_+$. Let $\mathfrak{L}(\mathfrak{A})$ be the class of (lower-semicontinuous) functions g on S which are pointwise supreme of increasing sequences $\{g_n\}$ in $\mathcal{S}(\mathcal{Q})$ for which $g_n \cdot \hat{\mu} \leqslant \hat{\gamma}$ for some $\hat{\gamma} \in L^1_+$ and all n.

THEOREM 8.13. *Let* $(\mathcal{Q}, \mu)$ *be a Boolean measure algebra.*

(a) $\hat{\nu} \in L^1_+$ *iff there is a* $g \in \mathcal{L}(\mathcal{Q})$ *such that* $\hat{\nu}(f) = \mu^*(gf)$ *for all* $f \in L^\infty(\mathcal{Q})$.

(b) *If* ϕ *is a bounded positive linear functional on* $L^1(\mathcal{Q})$ *and* $\hat{\nu}(f) = \phi(\hat{\mu}_f)$, *then there is a* $g_0 \in L^\infty(\mathcal{Q})$ *such that* $\hat{\mu}_{g_0}(f) = \mu^*(gf) = \hat{\nu}(f)$ *for all* $f \in L^\infty(\mathcal{Q})$, *and* $\phi(\hat{\gamma}) = \hat{\gamma}(g_0)$ *for all* $\hat{\gamma} \in L^1(\mathcal{Q})$.

PROOF. (a) Suppose that $\hat{\nu} \in L^1_+$. Let $\mathcal{G} = \{g \in \mathcal{S}(\mathcal{Q}) : g \cdot \hat{\mu} \leqslant \hat{\nu}\}$. Choose $g_n \in \mathcal{G}$ such that $(g_n \cdot \hat{\mu})(1) \to \sup\{g \cdot \hat{\mu}(1) : g \in \mathcal{G}\}$. It is easy to see that $h_1, h_2 \in \mathcal{G}$ implies $h_1 \vee h_2 \in \mathcal{G}$ so we can suppose that $\{g_n\}$ is increasing. Let $g(x) = \sup g_n(x)$; then, $g \in \mathcal{L}(\mathcal{Q})$ and $\hat{\gamma}(f) = \mu^*(gf)$ is in $L^1(\mathcal{Q})$. Suppose $\hat{\nu} - \hat{\gamma} \neq 0$. Then by 8.11 there is an α such that $(\hat{\nu} - \hat{\gamma} - \alpha\hat{\mu})^+ \neq 0$ and hence an $a \in \mathcal{Q}$ such that $(\hat{\nu} - \hat{\gamma} - \alpha\hat{\mu})\chi(a)f \geqslant 0$ for all $f \geqslant 0$ by 8.12. But then, for $f \geqslant 0$,

$$(g_n + \alpha\chi(a)) \cdot \hat{\mu}(f) \leqslant \hat{\gamma}(f) + \alpha\hat{\mu}_a(f) \leqslant \hat{\nu}(f) + (\hat{\nu} - \hat{\gamma})(\chi(a) - 1)f \leqslant \hat{\nu}(f)$$

so that $g_n + \alpha\chi(a) \in \mathcal{G}$ for all n. But this implies $\mu(a) = \hat{\mu}(\chi(a)) = 0$ a contradiction.

Hence $\hat{\nu} = g \cdot \hat{\mu}$ and $\|\hat{\nu} - g_n \cdot \hat{\mu}\|_1 = \hat{\nu}(1) - (g_n \cdot \hat{\mu})(1) \to 0$.

(b) Amend the definition of $\mathcal{G}$ so that $g \in \mathcal{G}$ entails $\|g\| \leqslant \|\phi\|$. Since $\mathcal{Q}$ is σ-complete the lattice supremum $g_0 = \vee g_n \in L^\infty(\mathcal{Q})$ [11, 43D] and $g_0 \cdot \mu = \lim g_n \cdot \hat{\mu} = \hat{\nu}$ by 5.7 and 8.7.

We close now with some results on β-completeness in $L^\infty(\mathcal{Q})$ and weak* completeness in $L^1(\mathcal{Q})$.

THEOREM 8.14. $L^\infty(\mathcal{Q})$ *is* β*-complete for a Boolean measure algebra* $(\mathcal{Q}, \mu)$.

PROOF. We first show that $L^\infty(\mathcal{Q})$ is β-quasi-complete. By 8.5, β is metrizable on the set B_r. If $\{f_n\}$ is β-Cauchy, then the net $\{f_n - f_m\}_{(n,m) \in \mathfrak{N} \times \mathfrak{N}}$ is β-convergent to zero. By 8.6, $\{P(f_n - f_m)\}$ is convergent to zero in $L^1(\mathcal{Q})$. Hence $\{f_n \cdot \hat{\mu}\}$ is Cauchy in $L^1(\mathcal{Q})$ and so has a limit $\hat{\nu} \in L^1(\mathcal{Q})$. Letting $T(pg) = g \cdot \hat{\nu}$ we see that

$$\|T(Pg)\|_1 = \lim \|gf_n \cdot \hat{\mu}\|_1 \leqslant r\|g \cdot \hat{\mu}\|_1 = r\|Pg\|_1.$$

So by 8.10, $\hat{\nu} = f \cdot \hat{\mu}$, where $f = T'1 \in L^\infty$. Since $\|T\| \leqslant r$, we have $\|f\| \leqslant r$ and now by 8.8(c), $f_n \to^\beta f$.

That $L^\infty(\mathcal{Q})$ is β-complete now follows from a result of Raikov [21] because of 3.2 just as in [26, 7.2].

ADDED IN PROOF. The proof above illustrates a direct and close connection between β-completeness and the multiplier and Radon-Nikodým results just above. Alternatively, completeness can be found on §7 (6.3), for more generally,

THEOREM 8.15. *If $\mathfrak{A}$ is σ-complete and β is Hausdorff, then the completion of* $L^\infty(\mathfrak{A})_\beta$ *is* $L^1(\mathfrak{A})'$.

PROOF. A. Grothendieck's completeness theorem allows one to identify the completion of $L^\infty(\mathfrak{A})_\beta$ with the collection of linear forms ϕ on $L^1(\mathfrak{A})$ which are $\sigma(L^1, L^\infty)$ continuous on β-equicontinuous sets [22, p. 106]. Thus any $\phi \in L^1(\mathfrak{A})'$ lies in this completion, for by 7.6 the topologies $\sigma(L^1, L^\infty)$ and $\sigma(L^1, L^{1'})$ agree on β-equicontinuous sets. Conversely if ϕ is in the completion and ϕ fails to be bounded on $L^1(\mathfrak{A})$, say $|\phi(\hat{\mu})_n)| \geqslant n^2$, then $H = \{\hat{\mu}_n/n: n = 1, 2, \ldots\} \cup \{0\}$ is β-equicontinuous, yielding a contradiction.

Note also that for complete algebras, limits of β-Cauchy nets lying in sets B_r can be directly found using 6.5, 8.14, 6.7 and 6.8.

We have enough tools now to outline a generalization of an old argument [9, pp. 158, 290].

THEOREM 8.16. $L^1(\mathfrak{A})$ *is weak* sequentially complete.*

PROOF. If $\{\hat{\mu}_n\}$ is weak* Cauchy it is pointwise bounded on the Banach space $L^\infty(\mathfrak{A})$ and so uniformly bounded. Letting $\mathfrak{B} = \mathfrak{A}/\mathfrak{M}$, where $\mathfrak{M} = \bigcap |\mu_n|^{-1}(0)$, and $\hat{\nu} = \sum |\hat{\nu}_n|/2^n$ where $T'\hat{\nu}_n = \hat{\mu}_n$ (7.2(b)). It follows that $\mathfrak{B}$ is a Boolean measure algebra and $\{\hat{\nu}_n\}$ is weak* Cauchy in $L^1(\mathfrak{B})$.

We can then suppose (using 7.5) that $\mathfrak{A}$ is a measure algebra and know that $\hat{\nu}(f) = \lim \hat{\mu}_n(f)$ is a bounded functional on $L^\infty(\mathfrak{A})$. According to 3.3 and 8.4 we need to show that $\hat{\nu}$ is d-continuous on the complete (8.14) metric space $(B_r, \beta) = (B_r, d)$. Let

$$A_k = \bigcap_{n,m \geqslant k} \{f \in B_r: |\hat{\mu}_m(f) - \hat{\mu}_n(f)| \geqslant \varepsilon\},$$

each of which is d-closed. So at least one A_k has nonempty d-interior and a standard argument shows that there is then a $\delta > 0$ such that $d(f, 0) < \delta, f \in B_r$, implies $|\hat{\nu}(f)| \subset \varepsilon$, completing the proof.

It is interesting to note that 8.16 could be given the same proof as its analogue in [26], by using 4.9, once reduced to the case that β be Hausdorff using 7.2.

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